

Quantum Grassmannians and Normal Elements in Noetherian Rings

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Abstract

The main idea running throughout the thesis is that of a normal element. An element in a ring is normal if the one sided ideal it generates is actually two sided.

In the first half of the thesis (Chapters 2, 3 and 4), our aim is to study certain quantum coordinate algebras. We are particularly interested in two classes of subalgebras of the coordinate ring of quantum matrices, namely quantum Grassmannians and quantum Flag varieties. A basis is constructed for each of these algebras, from which we calculate their Gelfand-Kirillov dimensions.

A well known result in the classical theory is that the dehomogenisation of the coordinate ring of the $m \times n$ Grassmannian at the ‘rightmost’ minor is isomorphic to the coordinate ring of the $m \times (n - m)$ matrices. The commutative notion of dehomogenisation does not immediately pass over to non-commutative theory. However, in the graded case, we show that it is possible to define non-commutative dehomogenisation at a regular normal homogeneous element, x , of degree 1 by considering a subring of the localisation of the ring at that element. The relationship between the prime spectrum of the original ring and that of the dehomogenisation is considered and a homeomorphism between the graded primes in the ring (not containing x) and a certain subset of primes in the dehomogenisation is constructed. Turning our attention back to quantum Grassmannians, we obtain the desired result, that the dehomogenisation of the quantum Grassmannian at the ‘rightmost’ minor is isomorphic to the quantum matrices.

The smallest instructive example of a quantum Grassmannian is the 2×4 quantum Grassmannian $G_q(2, 4)$, and we restrict our attention to this case. By presenting the algebra as a factor ring of an iterated Ore extension, we see that $G_q(2, 4)$ is Auslander Gorenstein and Cohen Macaulay. We conjecture that this is also true in the general case. Finally, we consider the graded prime spectrum of $G_q(2, 4)$. The homeomorphism mentioned above obtains for us a correspondence between the graded primes in $G_q(2, 4)$ and a subset of the prime spectrum of 2×2 quantum matrices, the latter of which is known. Therefore, we can obtain those graded primes in $G_q(2, 4)$ not containing the ‘rightmost’ minor. Using the classical cell decomposition of a projective space as inspiration, we then sequentially deho-

mogenise and factorise the algebra, allowing us to completely describe its graded prime spectrum.

Van Oystaeyen and Li Huishi have shown that given a noetherian ring R and a regular central element $x \in R$, if the factor ring R/Rx and the localisation R_x are Auslander Regular, then R is Auslander Regular. The aim in Chapter 5 is to relax the condition of x being central in this result, to x being normal. This is achieved by considering methods and results used to prove the original result and ‘twisting’ them to meet our needs.

Finally, in Chapter 6, we study the Krull dimension of q -skew polynomial rings over commutative rings of finite Krull dimension. It is well known that the Krull dimension of a skew polynomial ring over a ring R is either equal to that of R , or exceeds that of R by exactly 1. We establish a criterion on the maximal ideals of R which will determine which of the two possibilities holds for a given q -skew polynomial ring over a commutative ring of finite Krull dimension.

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Chapter 1

Preliminaries

In this chapter, we will cover some basic results regarding

- noetherian rings and modules,
- graded rings and modules,
- localisation at Ore sets,
- Gelfand-Kirillov Dimension and
- homological algebra, in particular homological dimensions,

which will be required in the remainder of the thesis. The material appearing here is for the most part well known and thus references are given in place of proofs unless the proof is felt to be particularly instructive.

Throughout the thesis, rings and modules will be associative, rings will possess an identity element 1_R and modules will be unital. The reader should note that a definition or result given for right (left) modules will automatically be assumed for left (right) modules.

General references for what follows are [23] and [33].

1.1 Noetherian Rings and Modules

Definition 1.1.1. Let R be a ring and M be a right R -module.

(i) The R -module M is **finitely generated** if there is a finite subset $\{m_1, \dots, m_n\}$ of M such that M is generated by $\{m_1, \dots, m_n\}$, so

$$M = \sum_{i=1}^n m_i R = \{m_1 r_1 + m_2 r_2 + \dots + m_n r_n \mid r_i \in R\}.$$

In particular, M_R is **cyclic** if there exists an element m in M such that $M = mR$.

(ii) The module M satisfies the **ascending chain condition** (ACC) if every infinite chain of submodules of M

$$M_1 \subseteq M_2 \subseteq M_3 \subseteq \dots$$

has only finitely many distinct terms.

(iii) The module M is **noetherian** if every nonempty collection \mathcal{S} of submodules of M contains a maximal member; that is, \mathcal{S} contains a submodule N such that there is no submodule T of M in \mathcal{S} which properly contains N . Thus

$$N \text{ is maximal} \Leftrightarrow \text{if } T \in \mathcal{S} \text{ and } N \subseteq T, \text{ then } T = N.$$

Note: M_R is cyclic if and only if $M \cong R/I$ for some right ideal I of R .

Proposition 1.1.2. *Let R be a ring and M be a (right or left) R -module. The following are equivalent:*

- (i) M is noetherian;
- (ii) M satisfies the ascending chain condition;
- (iii) all submodules of M are finitely generated.

Proof. See [23] Proposition 1.1. □

Definition 1.1.3. A ring R is **right (left) noetherian** if it is noetherian as a right (left) R -module. We say that R is **noetherian** if it is both right and left noetherian.

Prime and Semiprime Ideals

Definition 1.1.4. Let R be a ring. A proper ideal P of R is a **prime ideal** if whenever I, J are ideals of R with $IJ \subseteq P$, then either $I \subseteq P$ or $J \subseteq P$.

A **prime ring** is a ring in which 0 is a prime ideal.

We will denote the set of prime ideals of R by $\text{Spec}(R)$.

Proposition 1.1.5. *Let P be a proper ideal in a ring R . Then the following are equivalent:*

- (i) P is a prime ideal;

- (ii) R/P is a prime ring;
- (iii) if I, J are right ideals of R such that $IJ \subseteq P$, then $I \subseteq P$ or $J \subseteq P$;
- (iv) if I, J are left ideals of R such that $IJ \subseteq P$, then $I \subseteq P$ or $J \subseteq P$;
- (v) if $r, s \in R$ with $rRs \subseteq P$, then either $r \in P$ or $s \in P$.

Proof. See [23] Proposition 2.1. □

Thus, when R is commutative, the definition above coincides with the usual commutative definition of a prime ideal.

Definition 1.1.6. An ideal M of R is a **maximal ideal** if it is a maximal element in the family of proper ideals of R .

Definition 1.1.7. A prime ideal of R is a **minimal prime** if it does not contain any other prime ideal.

Proposition 1.1.8. A maximal ideal of a ring R is also a prime ideal of R .

Proof. See [23] Proposition 2.2. □

Definition 1.1.9. An ideal I of a ring R is **semiprime** if it is equal to an intersection of prime ideals of R .

A **semiprime ring** is a ring in which 0 is a semiprime ideal.

Thus an ideal I is semiprime if and only if R/I is a semiprime ring.

Definition 1.1.10. The **prime radical** of a ring R is the intersection of all prime ideals of R , and is denoted by $N(R)$.

We immediately obtain the following result.

Proposition 1.1.11. A ring R is semiprime if and only if $N(R) = 0$. □

Skew Polynomial Rings and Skew Laurent Extensions

Definition 1.1.12. Let R be a ring and $\sigma : R \rightarrow R$ be an automorphism of R . A **σ -derivation** on R is an additive map $\delta : R \rightarrow R$ such that

$$\delta(rs) = \sigma(r)\delta(s) + \delta(r)s \text{ for all } r, s \in R.$$

We call the pair (σ, δ) a **skew derivation** on R . Given a skew derivation (σ, δ) on R , we can construct the **skew polynomial ring** $S := R[x; \sigma, \delta]$ with elements of the form $\sum r_i x^i$ and multiplication satisfying the relation

$$xr = \sigma(r)x + \delta(r) \quad \text{for all } r \in R;$$

the reader is referred to [23] for precise details of the construction.

Notes

1. Above we have taken σ to be an automorphism of the ring. In fact it is enough for σ to be an endomorphism; however, this degree of generality will never be required in this thesis.

2. The skew polynomial ring defined above is in fact a left skew polynomial ring. There is of course a similar construction for right skew polynomial rings with elements of the form $\sum x^i r_i$. However, we make the choice to use the left version whenever constructing a skew polynomial ring.

3. When $\delta = 0$ we denote the skew polynomial ring by $R[x; \sigma]$, and similarly when $\sigma = 1$ we denote the skew polynomial ring by $R[x; \delta]$.

4. The ring $S = R[x; \sigma, \delta]$ is often referred to as an **Ore extension** of R .

Definition 1.1.13. An **iterated skew polynomial ring** (or **iterated Ore extension**) is a ring of the form

$$R[x_1; \sigma_1, \delta_1][x_2; \sigma_2, \delta_2] \dots [x_n; \sigma_n, \delta_n]$$

where (σ_1, δ_1) is a skew derivation on R , whilst (σ_2, δ_2) is a skew derivation on $R[x_1; \sigma_1, \delta_1]$, etc.

Let $S = R[x; \sigma, \delta]$ be a skew polynomial ring. Then every nonzero element of S can be written uniquely in the form

$$s = \sum_{i=0}^n r_i x^i \quad \text{with } r_n \neq 0.$$

Definition 1.1.14. We call r_n the **leading coefficient** of s and say that s has **degree** n and write $\deg(s) = n$. If s is the zero polynomial, we say that s has leading coefficient 0 and $\deg(s) := -1$. Let I be a right ideal of S and let

$$I_n = \{s \in I \mid \deg(s) \leq n\}$$

and

$$\lambda_n(I) = \{r_n \mid r_n \text{ is the leading coefficient of some } s \in I_n\}.$$

Then $\lambda_n(I)$ is a right ideal of R , called the n th **leading right ideal** of I .

The $\lambda_n(I)$ in Definition 1.1.14 form an ascending chain of right ideals of R ; that is, $\lambda_n(I) \subseteq \lambda_{n+1}(I)$, and it is this property which is used to prove the noetherian result in the following Theorem ([33] Theorem 1.2.9).

Theorem 1.1.15. *Let $S = R[x; \sigma, \delta]$ be a skew polynomial ring. Then*

- (i) if σ is injective and R is an integral domain, then S is an integral domain;*
- (ii) if σ is an automorphism and R is a prime ring, then S is a prime ring;*
- (iii) if σ is an automorphism and R is right (left) noetherian, then S is right (left) noetherian.*

Proof. See [33] Theorem 1.2.9. □

Let R be a ring with an automorphism σ and let x be an indeterminate. Then we can construct a ring T such that T is a free left R -module with a basis $1, x, x^{-1}, x^2, x^{-2}, \dots$ and multiplication given by

$$xr = \sigma(r)x \text{ for all } r \in R;$$

again, the reader is referred to [23] for exact details of the construction.

Definition 1.1.16. The ring $T = R[x, x^{-1}; \sigma]$ is called a **skew Laurent extension** of R or a **skew Laurent ring**.

Theorem 1.1.17. *Let R be a ring, σ be an automorphism of R and let $T = R[x, x^{-1}; \sigma]$. Then,*

- (i) if R is an integral domain, T is an integral domain;*
- (ii) if R is a prime ring, then T is a prime ring;*
- (iii) if R is right (left) noetherian, then T is right (left) noetherian.*

Proof. See [33] Theorem 1.4.5. □

1.2 Graded Rings and Modules

Let G be an additive group with identity 1_G .

Definition 1.2.1. Let R be a ring. We say that R is **(G -)graded** if there is a family of additive subgroups $\{R_g \mid g \in G\}$ such that

$$R = \bigoplus_{g \in G} R_g \text{ and } R_g R_h \subseteq R_{g+h} \text{ for all } g, h \in G.$$

Definition 1.2.2. Let R be a G -graded ring and let M be a right R -module. Then M is a **(G -)graded** right R -module if there exist additive subgroups $\{M_g \mid g \in G\}$ such that

$$M = \bigoplus_{g \in G} M_g \text{ and } M_g R_h \subseteq M_{g+h} \text{ for all } g, h \in G.$$

For any $m \in M$, there is a unique decomposition $m = \sum_{g \in G} m_g$ with all but finitely many m_g non-zero; we call the m_g the **graded components** of m . If $m = m_g$ for some $g \in G$, we say that m is **homogeneous** of degree g .

A **graded submodule** of M is a submodule N such that $N = \bigoplus_{g \in G} (N \cap R_g)$.

Equivalently, a graded submodule of M is a submodule N such that, if $n \in N$, then all the graded components of n are also in N .

In particular, a **graded ideal** of R is an ideal I of R such that $I = \bigoplus_{g \in G} (I \cap R_g)$.

Definition 1.2.3. A **graded prime** ideal of R is a proper graded ideal P , such that whenever A, B are graded ideals of R with $AB \subseteq P$, then $A \subseteq P$ or $B \subseteq P$.

We denote the set of graded prime ideals of R by $\text{GrSpec}(R)$.

Definition 1.2.4. Let $R = \bigoplus_{i \in \mathbb{Z}} R_i$ be a \mathbb{Z} -graded ring. If $R_i = 0$ for all $i < 0$, we say that R is **\mathbb{N} -graded**.

Let $R = \bigoplus_{i \geq 0} R_i$ be an \mathbb{N} -graded ring. Then the ideals R and $R_+ := \bigoplus_{i \geq 1} R_i$ are called the **irrelevant** ideals of R , while all other ideals are called **relevant**.

Proposition 1.2.5. *Let R be a \mathbb{Z} -graded ring. A graded ideal P is prime if and only if for homogeneous elements $a, b \in R$ such that $aRb \subseteq P$, it follows that $a \in P$ or $b \in P$.*

Proof. See [35] Proposition II.1.4. □

Corollary 1.2.6. *Let R be a \mathbb{Z} -graded ring. An ideal P of R is a graded prime ideal if and only if P is a prime ideal which is graded; that is,*

$$\text{GrSpec}(R) = \{P \in \text{Spec}(R) \mid P \text{ is graded}\}.$$

Proof. Clearly $\{P \in \text{Spec}(R) \mid P \text{ is graded}\} \subseteq \text{GrSpec}(R)$.

Suppose $P \in \text{GrSpec}(R)$. We must show that P is prime. Let $a, b \in P$ be homogeneous elements such that $aRb \subseteq P$. Then

$$\begin{aligned} aRb \subseteq P &\Rightarrow (aR)(bR) \subseteq P \\ &\Rightarrow aR \subseteq P \text{ or } bR \subseteq P \\ &\Rightarrow a \in P \text{ or } b \in P. \end{aligned}$$

Therefore, by Proposition 1.2.5, P is prime. \square

For the most part, the rings which we will encounter will be \mathbb{N} or \mathbb{Z} -graded.

1.3 Gelfand-Kirillov dimension

Let k be a field and recall that a k -algebra is **affine** if it is finitely generated as an algebra. The material covered in this section can, for the most part, be found in [26].

Throughout this thesis we will only be concerned with the Gelfand-Kirillov dimension of finitely generated k -algebras, and therefore we only deal with the definition for affine k -algebras.

Let k be a field and A be an affine k -algebra with generating set $\{a_1, a_2, \dots, a_m\}$. We say that a finite dimensional subspace V of A is a **finite dimensional generating subspace** for A , if every element in A can be written as a linear combination of monomials formed by elements of V . For example, we could take V to be the k -vector space spanned by a_1, a_2, \dots, a_m , since every element of A can be written as a linear combination of monomials formed with a_1, a_2, \dots, a_m . Let $V^0 = k$ and V^n be the k -subspace spanned by monomials of the form $a_{i_1}^{s_1} a_{i_2}^{s_2} \dots a_{i_m}^{s_m}$, where $a_{i_j} \in \{a_1, a_2, \dots, a_m\}$ and $\sum s_i = n$. Then

$$A_n := V^0 + V^1 + \dots + V^n \quad \text{and} \quad A = \bigcup_{n=0}^{\infty} A_n.$$

Notice that if $1 \in V$, then $A_n = V^n$. Define $d_V(n) = \dim_k(A_n)$.

Definition 1.3.1. The **Gelfand-Kirillov dimension** of the finitely generated k -algebra A is

$$\text{GKdim}(A) := \overline{\lim} \left(\frac{\log d_V(n)}{\log n} \right),$$

where V is a finite dimensional generating subspace of A .

Note that the Gelfand-Kirillov dimension of A is independent of the choice of V (see [26] for details).

Of course, we have that $\log_n d_V(n) = \log d_V(n) / \log n$ and thus

$$\text{GKdim}(A) = \overline{\lim} \log_n d_V(n).$$

In calculating Gelfand-Kirillov dimension we will need the following result.

Lemma 1.3.2.

(i) If $0 \neq f \in \mathbb{Q}[x]$ is a polynomial of degree d , then there exist $a_0, a_1, \dots, a_d \in \mathbb{Q}$, with $a_d \neq 0$, such that

$$f(n) = a_d \binom{n}{d} + a_{d-1} \binom{n}{d-1} + \dots + a_1 \binom{n}{1} + a_0,$$

for all $n \in \mathbb{Z}$.

(ii) For a function $f : \mathbb{N} \rightarrow \mathbb{Q}$, the following are equivalent:

(a) there exist $a_0, a_1, \dots, a_d \in \mathbb{Q}$ and $m \geq 0$ such that for all $n \geq m$,

$$f(n) = a_d \binom{n}{d} + a_{d-1} \binom{n}{d-1} + \dots + a_1 \binom{n}{1} + a_0;$$

(b) there exist $a_1, a_2, \dots, a_d \in \mathbb{Q}$ and $m \geq 0$ such that for all $n \geq m$,

$$f(n+1) - f(n) = a_d \binom{n}{d-1} + a_{d-1} \binom{n}{d-2} + \dots + a_1.$$

Proof. See [26] Lemma 1.5. □

We now give some results concerning Gelfand-Kirillov dimension of related algebras.

Lemma 1.3.3. *Let A be a (finitely generated) k -algebra and B be a subalgebra of A . Then $\text{GKdim}(B) \leq \text{GKdim}(A)$.*

Proof. See [26] Lemma 3.1. □

The two results that follow appear in [24], the proofs are given here to demonstrate how one may find the Gelfand-Kirillov dimension of an algebra.

Lemma 1.3.4. *Let A be an affine k -algebra and let $B = A[x; \sigma, \delta]$ be an Ore extension of A . Then*

$$\text{GKdim}(B) \geq \text{GKdim}(A) + 1.$$

Proof. Let V be a finite dimensional generating subspace of A such that $1_A \in V$. Then $W = kx \oplus V$ is a finite dimensional generating subspace of B . Now

$$V^n \oplus V^n x \oplus V^n x^2 \oplus \dots \oplus V^n x^n \subseteq W^{2n}$$

and so .

$$n\dim(V^n) \leq \dim(W^{2n}).$$

Therefore

$$\begin{aligned} \text{GKdim}(B) &\geq \overline{\lim} \log_n d_W(2n) \\ &\geq \overline{\lim} \log_n n\dim(V^n) \\ &= \overline{\lim} \log_n n + \overline{\lim} \log_n \dim(V^n) \\ &= 1 + \text{GKdim}(A). \end{aligned}$$

□

Proposition 1.3.5. *Let A be a (finitely generated) k -algebra and let $B = A[x; \sigma, \delta]$ be an Ore extension of A . Suppose that A has a finite dimensional generating subspace V such that $1_A \in V$ and $\sigma(V) \subseteq V$. Then*

$$\text{GKdim}(B) = \text{GKdim}(A) + 1.$$

Proof. Let $W = kx + V$. Then W is a finite dimensional generating subspace of B .

We have that $\delta(V)$ is finite dimensional and since $A = \cup V^m$, there exists an integer $p \geq 0$ such that $\delta(V) \subseteq V^p$. We claim that

$$\delta(V^n) \subseteq V^{n+p} \text{ for all } n.$$

For $n = 0$ it is trivially true. Suppose $n \geq 1$ and note that by induction, $\delta(V^{n-1}) \subseteq V^{n-1+p}$. Then

$$\begin{aligned} \delta(V^n) &= \delta(V^{n-1}V) \\ &\subseteq \sigma(V^{n-1})\delta(V) + \delta(V^{n-1})V \\ &\subseteq V^{n-1}V^{1+p} + V^{n-1+p}V \\ &= V^{n+p}. \end{aligned}$$

We also claim that

$$\begin{aligned} W^n &\subseteq V^{pn} + V^{pn}x + \dots + V^{pn}x^n \\ &= \sum_{i=0}^n V^{pn}x^i \end{aligned}$$

for all $n \in \mathbb{N}$. To show this inclusion we use induction on n .

For $n = 0$ it is trivially true. Suppose $n \geq 0$. Then

$$\begin{aligned}
W^{n+1} &= WW^n \\
&\subseteq (kx + V) \left(\sum_{i=0}^n V^{pn} x^i \right) \\
&= \sum_{i=0}^n x V^{pn} x^i + \sum_{i=0}^n V^{pn+1} x^i \\
&\subseteq \sum_{i=0}^n [\sigma(V^{pn})x + \delta(V^{pn})]x^i + \sum_{i=0}^n V^{pn+1} x^i \\
&\subseteq \sum_{i=0}^n [V^{pn}x + V^{pn+p}]x^i + \sum_{i=0}^n V^{pn+1} x^i \\
&\subseteq \sum_{i=0}^n V^{pn} x^{i+1} + \sum_{i=0}^n V^{p(n+1)} x^i \\
&\subseteq \sum_{i=0}^{n+1} V^{p(n+1)} x^i.
\end{aligned}$$

Thus,

$$\dim(W^n) \leq (n+1) \dim(V^{pn}).$$

Let $U = V^p$. Then U is a finite dimensional generating set for A and $d_W(n) \leq (n+1)d_U(n)$. Therefore

$$\begin{aligned}
\text{GKdim}(B) &= \overline{\lim} \log_n d_W(n) \\
&\leq \overline{\lim} \log_n \{(n+1)d_U(n)\} \\
&= \overline{\lim} \log_n(n+1) + \overline{\lim} \log_n d_U(n) \\
&= 1 + \text{GKdim}(A)
\end{aligned}$$

and Lemma 1.3.4 gives us the required equality. \square

We now extend the definition of Gelfand-Kirillov dimension to modules over affine k -algebras. Note that we will only be concerned with finitely generated modules and thus only the definition for such modules appears here.

Let A be an affine k -algebra with finite dimensional generating subspace V such that $1 \in V$. Let M be a finitely generated right A -module with finite dimensional subspace F , which generates M as an A -module. Then

$$M = \cup_{n=0}^{\infty} FV^n.$$

Define $d_{V,F}(n) = \dim_k(FV^n)$.

Definition 1.3.6. The Gelfand-Kirillov dimension of M is

$$\text{GKdim}(M) := \overline{\lim} \log_n d_{V,F}(n).$$

Note that the Gelfand-Kirillov dimension of M is independent of the choice of V and F (see [26] for details).

Clearly,

$$\text{GKdim}(A) = \text{GKdim}(A_A);$$

that is, the Gelfand-Kirillov dimension of A as an algebra coincides with the Gelfand-Kirillov dimension of A as a right A -module. The following proposition considers the Gelfand-Kirillov dimension of some related modules.

Proposition 1.3.7. *Let A be an affine k -algebra and M, M_1, \dots, M_n be finitely generated right A -modules.*

- (i) *Suppose $M = M_1 \oplus \dots \oplus M_n$. Then $\text{GKdim}(M) = \max_i \{\text{GKdim}(M_i)\}$.*
- (ii) *Let $0 \rightarrow K \rightarrow M \rightarrow L \rightarrow 0$ be an exact sequence of right A -modules. Then $\text{GKdim}(M) \geq \max\{\text{GKdim}(K), \text{GKdim}(L)\}$.*
- (iii) *If $MI = 0$ for some ideal I of A , then $\text{GKdim}(M_A) = \text{GKdim}(M_{A/I})$.*
- (iv) $\text{GKdim}(M) \leq \text{GKdim}(A)$.
- (v) *Let $\alpha : M \rightarrow M$ be an injective module homomorphism. Then $\text{GKdim}(M/\alpha(M)) \leq \text{GKdim}(M) - 1$.*
- (vi) *Suppose $M = M_1 + \dots + M_n$. Then $\text{GKdim}(M) = \max_i \{\text{GKdim}(M_i)\}$.*

Proof. See [26] Proposition 5.1. □

Definition 1.3.8. Let M be a right A -module and let $x \in A$. We say that x is a **non-zero divisor** on M if whenever $mx = 0$, we have $m = 0$.

From Proposition 1.3.7 (v), we have the immediate corollary.

Corollary 1.3.9. *Let A be an affine k -algebra and M be a right A -module. Suppose that $x \in A$ is central and x is a non-zero divisor on M . Then*

$$\text{GKdim}(M/Mx) \leq \text{GKdim}(M) - 1.$$

□

Of course, in non-commutative algebra there is no guarantee that a central element will exist; the ‘next best thing’ is a normal element. Recall that $x \in R$ is **normal** if it generates a two sided ideal; that is, $x \in R$ is normal if $xR = Rx$. In the above Corollary it is possible, with a little more work, to relax the condition of x being central to x being normal.

Proposition 1.3.10. *Let A be an affine k -algebra and M be a right A -module. Suppose that $x \in A$ is normal and is a non-zero divisor on M . Then*

$$\text{GKdim}(M/Mx) \leq \text{GKdim}(M) - 1.$$

Proof. This proof is adapted from [26] Proposition 3.15.

Note that Mx is a submodule of M due to the normality of x . Define $\overline{M} := M/Mx$.

Let V be a finite dimensional generating subspace of A such that $1_A, x \in V$, and let \overline{F} be a finite dimensional subspace of \overline{M} , which generates \overline{M} as an A -module. Then there is a finite dimensional subspace F of M which generates M as an A -module and has image \overline{F} under the canonical homomorphism $M \rightarrow M/Mx$. Then

$$M = \cup FV^n \text{ and } \overline{M} = \cup \overline{FV^n} = \cup \overline{F}V^n.$$

Now let D_n be a vector space complement of $FV^n \cap Mx$ in FV^n ; that is,

$$(FV^n \cap Mx) \oplus D_n = FV^n$$

as vector spaces. Then

$$\overline{F}V^n = \frac{FV^n + Mx}{Mx} \cong \frac{FV^n}{FV^n \cap Mx} \cong D_n$$

as vector spaces. Now, since $D_n \cap Mx = 0$, the sum

$$D_n + D_nx + D_nx^2 + \dots + D_nx^n$$

is direct. Also,

$$FV^{2n} \supseteq D_n \oplus D_nx \oplus D_nx^2 \oplus \dots \oplus D_nx^n$$

and thus

$$\dim(FV^{2n}) \geq n \dim(D_n) = n \dim(\overline{F}V^n).$$

Therefore

$$\begin{aligned}
 \text{GKdim}(M/Mx) + 1 &= \overline{\lim} \log_n d_{V,\overline{F}}(n) + 1 \\
 &= \overline{\lim} \log_n d_{V,\overline{F}}(n) + \lim_{n \rightarrow \infty} \log_n n \\
 &= \overline{\lim} \log_n d_{V,\overline{F}}(n) n \\
 &\leq \overline{\lim} \log_n d_V(2n) \\
 &\leq \text{GKdim}(M),
 \end{aligned}$$

as required. \square

In fact in the case that A is \mathbb{N} -graded and x is a normal homogeneous element of degree $d > 0$, we have equality in Proposition 1.3.10. In order to prove this we introduce the following definition.

Definition 1.3.11. Let R be a ring, M be a right R -module and $\sigma : R \rightarrow R$ be an automorphism of R . Then M^σ is the right R -module with the same underlying abelian group as M , but with multiplication given by

$$m *_\sigma r := m\sigma(r) \text{ for all } m \in M^\sigma.$$

To emphasise whether we are thinking of an element in M or in M^σ , we will sometimes give elements in M^σ a superscript σ .

The reader should note that this idea of ‘twisting’ the module map is extremely useful when working with normal elements and will be used extensively in later chapters.

Lemma 1.3.12. *Let R be a ring and x be a normal non-zero divisor in R . Let $\sigma : R \rightarrow R$ be the automorphism obtained by conjugating by x ; that is, $xr = \sigma(r)x$. Let M be a right R -module and let*

$$\mu : M^\sigma \rightarrow M$$

be right multiplication by x . The map μ is an R -module homomorphism.

Proof. Clearly μ is an additive map, we check that it is an R -module map. Let $r \in R$ and $m^\sigma \in M^\sigma$. Then

$$\begin{aligned}
 \mu(m^\sigma *_\sigma r) &= \mu(m\sigma(r)) \\
 &= (m\sigma(r))x \\
 &= m(\sigma(r)x) \\
 &= m(xr) \\
 &= (mx)r \\
 &= \mu(m)r.
 \end{aligned}$$

□

We will require the following result.

Lemma 1.3.13. *Let $A = \oplus_{i \in \mathbb{Z}} A_i$ be a finitely \mathbb{Z} -graded affine k -algebra (so each A_i is a finite dimensional vector space) and let $M = \oplus_{i \in \mathbb{Z}} M_i$ be a finitely graded, finitely generated right A -module. Let*

$$A(n) = \oplus_{i=-n}^n A_i \quad \text{and} \quad M(n) = \oplus_{i=-n}^n M_i$$

and define

$$d_A(n) = \dim_k(A(n)) \quad \text{and} \quad d_M(n) = \dim_k(M(n)).$$

Then,

$$\text{GKdim}(A) = \overline{\lim} \log_n d_A(n) \quad \text{and} \quad \text{GKdim}(M) = \overline{\lim} \log_n d_M(n).$$

Proof. See [26] Lemma 6.1. □

Lemma 1.3.14.

$$\text{GKdim}(M) = \text{GKdim}(M^\sigma)$$

□

A module M , as above, is said to have **polynomial growth** of degree s if there exist $C, m \in \mathbb{N}$ such that $d_M(n) \leq Cn^s$ and $n^s \leq d_M(mn)$ for almost all n . Note that the following proposition appears in [27] Lemma 5.7, with the restriction that M must have polynomial growth.

Proposition 1.3.15. *Let $A = \oplus_{i \in \mathbb{N}} A_i$ be a finitely \mathbb{N} -graded affine k -algebra and let $x \in A_d$, $d > 0$, be normal in A . Let $M = \oplus_{i \in \mathbb{Z}} M_i$ be a finitely graded, finitely generated non-zero right A -module such that x is a non-zero divisor in M . Then*

$$\text{GKdim}(M/Mx) = \text{GKdim}(M) - 1.$$

Proof. By Lemma 1.3.10, we have that $\text{GKdim}(M/Mx) \leq \text{GKdim}(M) - 1$ and we must show the reverse inequality. Let $0 \neq M = \oplus_{i \in \mathbb{Z}} M_i$. We may assume that $M_i = 0$ for all $i < p \leq 0$, and so $M = \oplus_{i \geq p} M_i$.

Let $\overline{M} = M/Mx$. Then $\overline{M}_i = M_i/M_{i-d}x$ gives a grading on \overline{M} . Since x is a non-zero divisor in M , we have exact sequences

$$0 \rightarrow M_{i-d}^\sigma \xrightarrow{\rho} M_i \rightarrow \overline{M}_i \rightarrow 0,$$

where ρ is right multiplication by x . We deduce that

$$d_{\overline{M}}(n) = d_M(n) - d_{M^\sigma}(n - d)$$

and since

$$d_{M^\sigma}(n - d) = d_M(n - d),$$

we have that

$$d_M(n) = d_{\overline{M}}(n) + d_M(n - d).$$

Note that $\overline{M_p} = M_p$ and if $i > (n - p)/d$, then $d_M(n - id) = 0$. Therefore

$$d_M(n) = \sum_{i=0}^{E(n)} d_{\overline{M}}(n - id),$$

where $E(n)$ is the integer part of $(n - p)/d$.

Certainly $E(n) \leq n$, and $d_{\overline{M}}(n - id) \leq d_{\overline{M}}(n)$ for $i = 0, \dots, E(n)$, so

$$d_M(n) \leq (n + 1) d_{\overline{M}}(n).$$

Therefore

$$\begin{aligned} \text{GKdim}(M) = \overline{\lim} \log_n d_M(n) &\leq \overline{\lim} \log_n \{(n + 1) d_{\overline{M}}(n)\} \\ &= \overline{\lim} \log_n(n + 1) + \overline{\lim} \log_n d_{\overline{M}}(n) \\ &= 1 + \text{GKdim}(\overline{M}); \end{aligned}$$

that is, $\text{GKdim}(M/Mx) \geq \text{GKdim}(M) - 1$ and we have equality, as required. \square

1.4 Localisation

Let R be a commutative ring and P be a prime ideal of R . The factor ring R/P is a commutative domain and thus has a field of fractions, K say. In this case the study of K can yield much information about R , and thus passing from R to the factor ring R/P and then to K has proved to be a useful technique in commutative ring theory. As one would expect, in the non-commutative case it does not necessarily work out quite as nicely. Let R be a non-commutative ring and let P be a prime ideal of R . It is not even clear, as yet, what one would mean by a ‘quotient field’ of R/P .

Recall the following definition.

Definition 1.4.1. A **multiplicative set** X of a ring R is any subset of the ring such that $1 \in X$ and X is closed under multiplication.

The problem above of finding a ‘quotient field’ of R/P is a special case of the general theory of non-commutative rings of fractions (or non-commutative localisation). Let us return for a moment to the commutative case. A well known, and indeed well used construction is the localisation of a commutative ring R at a multiplicatively closed set X (such that $1 \in X$). A new ring R_X is constructed consisting of fractions of the form r/x where $r \in R$ and $x \in X$ subject to the following equivalence relation:

$$r/x \sim r'/x' \Leftrightarrow \text{there exists } y \in X \text{ such that } (rx' - r'x)y = 0.$$

In this section, we summarise how this construction can be extended to the non-commutative case and look at the relationship between the prime spectrum of a noetherian ring and the prime spectrum of its localisation. The material contained in this section can be found in more detail in [23] Chapter 9 and to some extent Chapter 5.

In the commutative case we have the following result.

Lemma 1.4.2. *Let X be a multiplicative subset of the (commutative) ring R and let $\phi : R \rightarrow R_X$ be the ring homomorphism defined by $\phi(r) = r/1$. Then*

- (i) *for each $x \in X$, $\phi(x)$ is a unit in R_X ;*
- (ii) *$\ker \phi = \{r \in R \mid xr = 0 \text{ for some } x \in X\}$;*
- (iii) *if $\psi : R \rightarrow T$ is a ring homomorphism such that $\psi(x)$ is a unit in T for all $x \in X$, then ψ factors uniquely through ϕ ; that is, there exists $\nu : R_X \rightarrow T$ such that $\psi = \nu\phi$. \square*

The corresponding construction in the non-commutative case should ‘mimic’ the commutative theory in that our ring of fractions should satisfy the same kind of properties as those above. With this in mind, the following definition is made.

Definition 1.4.3. Given a multiplicative subset X of a ring R , a **(right) localisation** of R with respect to X is a ring homomorphism $\phi : R \rightarrow S$ such that

- (i) *for each $x \in X$, $\phi(x)$ is a unit in S ;*
- (ii) *$\ker \phi = \{r \in R \mid xr = 0 \text{ for some } x \in X\}$;*
- (iii) *each element of S has the form $\phi(r)\phi(x)^{-1}$ for some $r \in R$, $x \in X$.*

Definition 1.4.4. Let X be a multiplicative subset of R . Then

1. X is said to satisfy the **right Ore condition** if

$$rX \cap xR \neq \emptyset \text{ for all } r \in R \text{ and } x \in X;$$

2. X is said to be **right reversible** if

whenever $r \in R, x \in X$ such that $xr = 0$, then $\exists x' \in X$ such that $rx' = 0$;

3. X is called a **right denominator set** if it is a right reversible, right Ore set.

Lemma 1.4.5. *Let X be a multiplicative subset of R and suppose that there exists a localisation $\phi : R \rightarrow S$ of R with respect to X . Then X is a right denominator set.*

Proof. See [23] Lemma 9.1. □

Notice that in the definition of localisation we do not have a condition equivalent to that of Lemma 1.4.2 (iii). This is rectified in the following Proposition.

Proposition 1.4.6. *Let X be a right denominator set in R and suppose that there is a right localisation $\phi : R \rightarrow S$ for R with respect to X . If $\psi : R \rightarrow T$ is a ring homomorphism such that $\psi(x)$ is a unit in T , for all $x \in X$, then ψ factors uniquely through ϕ ; that is, there is a unique ring homomorphism $\nu : S \rightarrow T$ such that $\psi = \nu\phi$.*

Proof. See [23] Proposition 9.4. □

Corollary 1.4.7. *Let X be a right denominator set in a ring R . Suppose that $\phi_1 : R \rightarrow S_1$ and $\phi_2 : R \rightarrow S_2$ are right localisations of R with respect to X . Then there is a unique ring isomorphism $\eta : S_1 \rightarrow S_2$ such that $\eta\phi_1 = \phi_2$.*

Proof. See [23] Corollary 9.5. □

Lemma 1.4.5 establishes that X being a right denominator set is a necessary condition for the existence of a right localisation of R with respect to X . In fact it is necessary and sufficient.

Theorem 1.4.8. *Let X be a multiplicative subset of R . Then there exists a (right) localisation of R with respect to X if and only if X is a right denominator set.*

Proof. See [23] Theorem 9.7. □

Definition 1.4.9. Let M be a right R -module and $X \subseteq M$ be any subset of M . The **annihilator** of X is the right ideal

$$\text{ann}_R(X) = \{r \in R \mid xr = 0 \text{ for all } x \in X\}.$$

Of course, there is an equivalent definition for left modules and when $M = R$ we use the notation $\text{rann}_R(X)$ and $\text{lann}_R(X)$ to make it clear whether we are thinking of R as a right or left module over itself.

A **right annihilator** in R is any right ideal which is equal to the right annihilator of some subset of R .

Proposition 1.4.10. *Let X be a right Ore set in a ring R . If R satisfies the ascending chain condition on right annihilators of elements, then X is right reversible.*

Proof. See [23] Proposition 9.9. □

Corollary 1.4.11. *Let X be a multiplicative subset of a noetherian ring R . Then there exists a (right) localisation of R with respect to X if and only if X is a right Ore set.*

In the main body of this thesis the rings in which we are interested will be noetherian; thus for the existence of a right localisation of a ring at a set X , we need only check that the set satisfies the right Ore condition.

Let X be a right denominator set in a ring R . Then by Theorem 1.4.8 there exists a right localisation $\phi : R \rightarrow S$ of R with respect to X which by Corollary 1.4.7 is essentially unique. Henceforth, we will denote S by R_X . In a standard misuse of notation, we will refer to the ring R_X as the (right) localisation of R at X and to $\phi : R \rightarrow R_X$ as the **natural map**. We will also take the liberty of writing elements in the localisation as rx^{-1} rather than $\phi(r)\phi(x)^{-1}$, which can become rather cumbersome.

Goldie Theory

Let R be a noetherian prime ring. Among other things, Goldie Theory solves for us the problem of finding a right ‘quotient field’ of R , as discussed at the beginning of the section. Recall that a non-zero divisor in a ring R is also referred to as a **regular** element in R .

Definition 1.4.12. Let X be the set of regular elements in a ring R . Then if the localisation of R at X exists, we will say that it is the **right ring of fractions** of R .

Example 1.4.13. Let R be a noetherian domain. Then R has a right ring of fractions.

Proof. Since R is noetherian we need only check that $X = R \setminus \{0\}$ is a right Ore set in R . Let $x \in X$, $r \in R$ and suppose that

$$xR \cap rX = \emptyset.$$

Equivalently,

$$xR \cap rR = 0$$

and so the infinite sum

$$rR + xR + xrR + xr^2R + \dots + xr^nR + \dots$$

is direct, contradicting R being noetherian. □

We now give a brief summary of Goldie's Theorems, for which we require the following definitions.

Definition 1.4.14. Let R be a ring and M be a non-zero right R -module.

1. A non-zero submodule N of M is **essential** in M if the intersection of N with any other non-zero submodule of M is non-zero.
2. We say that M is **uniform** if every non-zero submodule of M is essential in M .
3. We say that M has **finite rank** if and only if M has an essential submodule which is a finite direct sum of uniform submodules. Equivalently M has finite rank if and only if M contains no infinite direct sums of non-zero submodules ([23] Theorem 4.13).

Definition 1.4.15. A right **Goldie ring** is a ring R which has finite rank (as a right R -module) and satisfies the ascending chain condition on right annihilators.

In particular, a right noetherian ring is right Goldie.

The following Proposition is the crucial result in proving Goldie's Theorem.

Proposition 1.4.16. *Let R be a semiprime right Goldie ring and I be a right ideal of R . The ideal I is essential in R if and only if I contains a regular element.*

Proof. See [23] Proposition 5.9. □

Theorem 1.4.17. *A ring R has a semisimple right ring of fractions if and only if R is a semiprime right Goldie ring.*

Proof. See [23] Theorem 5.10. □

Corollary 1.4.18. *A prime noetherian ring R has a semisimple right ring of fractions.* □

Lemma 1.4.19. *Suppose that a ring R has a right ring of fractions, S . Then S is a simple ring if and only if R is a prime ring.*

Proof. See [23] Lemma 5.11. □

Theorem 1.4.20. *A ring R has a simple artinian right ring of fractions if and only if R is a prime right Goldie ring.*

Proof. See [23] Theorem 5.12. □

Example 1.4.21. Let R be a noetherian domain. Then R has a division ring of fractions (cf. Example 1.4.13).

The following example is, for our purposes, the most important and will often be used in later chapters.

Example 1.4.22. Let R be a noetherian ring and $x \in R$ be a normal non-zero divisor in R . Then $X = \{x^i \mid i \geq 0\}$ is a right Ore set in R and therefore R_X , the localisation of R at X , exists. In fact we will call this localisation the localisation of R at x .

Proof. The set X is certainly a multiplicative subset of R . Let $r \in R$ and $x^n \in X$ for some $n \geq 0$. Since x is normal in R there exists $r' \in R$ such that

$$rx^n = x^n r'.$$

Therefore

$$rX \cap x^n R \neq \emptyset$$

and X is a right Ore set. □

Localisation of a module

Lemma 1.4.23. *Let R be a ring, X be a right Ore set in R and M be a right R -module. Then*

$$t_X(M) = \{m \in M \mid mx = 0 \text{ for some } x \in X\}$$

is a submodule of M . If X is also right reversible, then $t_X(R) = \ker(\phi)$ (where ϕ is the natural map of the localisation).

Proof. See [23] Lemma 9.3. □

Definition 1.4.24. The module $t_X(M)$ is called the **X -torsion submodule** of M . We say that M is **X -torsion** if $t_X(M) = M$ and that M is **X -torsionfree** if $t_X(M) = 0$.

Having defined the localisation of a ring at a set X , we now concern ourselves with the localisation of modules over the ring. The preceding theory would indicate that we should be considering ‘fractions’ with numerators from the module and denominators from the set X .

Let X be a right denominator set in a ring R and let M be a right R -module. A **localisation** of M with respect to X is a right R_X -module N together with a module homomorphism $f : M \rightarrow N$ such that

- (1) each element of N has the form $f(m)x^{-1}$ for some $m \in M$ and $x \in X$;
- (2) $\ker f = t_X(M)$.

The following proposition mirrors property (iii) of Lemma 1.4.2.

Proposition 1.4.25. *Let X be a right denominator set in a ring R and let M be a right R -module. Suppose there is a localisation of M with respect to X , $f : M \rightarrow N$. If T is a right R_X -module and $g : M \rightarrow T$ is an R -module homomorphism, then there is a unique R_X -module homomorphism $h : N \rightarrow T$, such that $g = hf$.*

Proof. See [23] Proposition 9.10. □

The following Corollary establishes the uniqueness of the localisation of a right R -module, if such a localisation exists.

Corollary 1.4.26. *Let X be a right denominator set in a ring R and let M be a right R -module. Suppose $f_1 : M \rightarrow N_1$ and $f_2 : M \rightarrow N_2$ are localisations of M*

with respect to X . Then there is a unique R_X -module isomorphism $g : N_1 \rightarrow N_2$ such that $gf_1 = f_2$.

Proof. See [23] Corollary 9.11. □

In fact if X is a right denominator set in R , then the localisation of a right R -module exists.

Theorem 1.4.27. *Let X be a right denominator set in R . Then for every right R -module M , there exists a localisation $f : M \rightarrow N$ of M .*

Proof. See [23] Theorem 9.13. □

Thus, given a right denominator set in a ring R and a right R -module M , there exists a localisation $f : M \rightarrow N$ of M , which, by Corollary 1.4.26, is essentially unique. In the usual abuse of notation, we will refer to the module N as the localisation of M at X and denote it by M_X . We will also simplify the notation by writing elements of M_X in the form mx^{-1} , rather than $f(m)x^{-1}$.

Proposition 1.4.28. *Let X be a right denominator set in a ring R and let M be a right R -module. Then the map*

$$\begin{aligned} g & : M \times R_X \rightarrow M_X \\ (m, s) & \mapsto (m1^{-1})s \end{aligned}$$

induces an R_X -module isomorphism from $M \otimes_R R_X$ onto M_X .

Proof. See [23] Proposition 9.14. □

Proposition 1.4.28 gives us an alternative definition of the localisation of a right R -module in terms of tensor products. In later chapters, this definition will be used without further comment.

Let X be a right denominator set in R and let M be a right R -module.

Let N be a submodule of M_X . The **contraction** of N to M is the set $\{m \mid m1^{-1} \in N\}$ and is sometimes denoted N^c . In the remainder of the thesis we will use the convenient notation $N \cap M$ to denote the contraction of N to M . The reader should note that this is merely a useful abuse of notation and that it is possible for the localisation to have a kernel.

Let T be a submodule of M . Then the **extension** of T to M_X is the set TR_X , which is sometimes denoted by T^e .

Proposition 1.4.29. *Let X be a right denominator set in R and let M be a right R -module.*

(i) *If N is an R_X -submodule of M_X , then $N \cap M$ is an R -submodule of M , the factor $M/(N \cap M)$ is X -torsion free and $N = (N \cap M) R_X$; that is, $N = N^{ce}$.*

(ii) *If T is an R -submodule of M , then TR_X is an R_X -submodule of M_X and $T \subseteq TR_X \cap M$. Also, $(TR_X \cap M)/T = t_X(M/T)$, and thus $T = TR_X \cap M$ (that is, $T = T^{ec}$) if and only if M/T is X -torsion free.*

In fact,

(iii) *contraction and extension are inverse lattice isomorphisms between the lattice of R_X -submodules of M_X and the lattice of R -submodules T of M such that M/T is X -torsion free.*

Proof. See [23] Theorem 9.17. □

Corollary 1.4.30. *Let X be a right denominator set in R and let M be a right R -module. Then*

(i) *if M is noetherian (artinian), then M_X is a noetherian (artinian) R_X -module;*

(ii) *if M is simple, then either $M_X = 0$ or M_X is a simple R_X -module.* □

In particular, Proposition 1.4.29 applies to ideals in localisations of rings. As promised, the following set of results establishes the relationship between the prime spectrum of a ring R and the prime spectrum of its localisation at a right denominator set.

Proposition 1.4.31. *Let X be a right denominator set in a ring R and let J be an ideal of R_X . Then*

(i) *$J \cap R$ is an ideal of R and $(J \cap R) R_X = J$;*

(ii) *if $J \cap R$ is a prime (semiprime) ideal in R , then J is a prime (semiprime) ideal in R_X .*

Proof. See [23] Proposition 9.19. □

Note that if I is an ideal of R , then IR_X need not be an ideal of R_X whilst if J is prime, then $J \cap R$ is not necessarily prime or even semiprime.

However, the rings which we will be interested in later will all be noetherian and the above note can be dealt with by Proposition 1.4.32. First recall that given an

ideal I of a ring R , an element $r \in R$ is said to be regular modulo I if the coset $r + I$ is regular in the ring R/I . The set of all elements in R which are regular modulo I is denoted by $\mathcal{C}(I)$.

Proposition 1.4.32. *Let X be a right denominator set in a ring R and suppose that R_X is noetherian. Then*

(i) *if I is an ideal of R , then IR_X is an ideal of R_X . Suppose that $(R/I)_R$ is X -torsion free. Then I is prime (semiprime) if and only if IR_X is prime (semiprime);*

(ii) *if J is an ideal of R_X , then J is prime (semiprime) if and only if $J \cap R$ is prime (semiprime);*

(iii) *if P is a prime (semiprime) ideal of R , then $P = Q \cap R$ for some prime (semiprime) ideal Q of R_X if and only if $X \subseteq \mathcal{C}(P)$.*

Proof. See [23] Theorem 9.20. □

Theorem 1.4.33. *Let X be a right denominator set in a noetherian ring R . Then contraction and extension are inverse bijections between the set of prime ideals of R_X and the set of those primes in R disjoint from X .*

Proof. See [23] Theorem 9.22. □

In particular, we have the following important example (cf. Example 1.4.22).

Example 1.4.34. Let R be a noetherian ring and let x be a normal non-zero divisor in R . Then contraction and extension are inverse bijections between the set of prime ideals in R_x , and the set of those primes in R not containing x .

1.5 Homological Algebra

Definition 1.5.1. A right R -module is **free** if it is a direct sum of copies of R .

An R -module P is **projective** if for R -modules B, C with a surjective R -module homomorphism $\beta : B \rightarrow C$

$$\begin{array}{ccc} & P & \\ \gamma \swarrow & \downarrow \alpha & \\ B & \xrightarrow{\beta} & C \longrightarrow 0 \end{array}$$

and any R -module map $\alpha : P \rightarrow C$, there is an R -module map $\gamma : P \rightarrow B$ such that the above diagram commutes; that is, $\alpha = \beta\gamma$.

Note that a module P is projective if and only if it is direct summand of a free module.

A **projective resolution** of a module M is an exact sequence

$$\dots \rightarrow P_{n+1} \xrightarrow{d_{n+1}} P_n \rightarrow \dots \rightarrow P_1 \xrightarrow{d_1} P_0 \rightarrow M \rightarrow 0$$

such that P_n is projective for all n . The **deleted resolution** is then

$$\dots \rightarrow P_{n+1} \xrightarrow{d_{n+1}} P_n \rightarrow \dots \rightarrow P_1 \xrightarrow{d_1} P_0 \rightarrow 0.$$

Note that no information is lost in removing M from the resolution, since $M \cong \text{coker}(d_1)$.

If there exists an integer n such that M has a projective resolution of the form

$$0 \rightarrow P_n \rightarrow P_{n-1} \rightarrow \dots \rightarrow P_1 \rightarrow P_0 \rightarrow M \rightarrow 0,$$

then the **projective dimension** of M , denoted $\text{pd}(M)$, is at most n . If n is the least such integer, then $\text{pd}(M) = n$. If no such finite integer exists, then $\text{pd}(M) = \infty$.

Clearly, M is projective if and only if $\text{pd}(M) = 0$.

Definition 1.5.2. A right R -module E is injective if for any R -modules $A \subseteq B$, every R -module map $f : A \rightarrow E$ can be extended to an R -module map $g : B \rightarrow E$, such that the following diagram commutes:

$$\begin{array}{ccccc} & & E & & \\ & & \uparrow f & \nearrow g & \\ 0 & \longrightarrow & A & \longrightarrow & B. \end{array}$$

An **injective resolution** of a module M is an exact sequence

$$0 \rightarrow M \rightarrow E^0 \rightarrow E^1 \rightarrow \dots \rightarrow E^n \rightarrow E^{n+1} \rightarrow \dots$$

where each E^i is an injective module.

One can go on in the manner of Definition 1.5.1 to define the deleted resolution and **injective dimension**, $\text{id}(M)$. We leave this as an exercise for the reader.

The following results establish some of the basic properties of the projective dimension of a module.

Proposition 1.5.3. Consider a short exact sequence of right R -modules,

$$0 \rightarrow M' \rightarrow M \rightarrow M'' \rightarrow 0.$$

Then $\text{pd}(M) \leq \max\{\text{pd}(M'), \text{pd}(M'')\}$.

Proof. See [33] 7.1.6. □

Proposition 1.5.4. Let x be a regular normal element of R and let M be a right R -module. If $M_{R/Rx}$ is non-zero and $\text{pd}_{R/Rx}(M) = n < \infty$, then $\text{pd}_R(M) = n+1$.

Proof. See [33] Theorem 7.3.5(i). □

Definition 1.5.5. Let R be a ring and let \mathcal{M}_R be the category of right R -modules. The (right projective) **global dimension** of R is given by

$$\text{gldim}(R) := \sup\{\text{pd}(A) \mid A \in \mathcal{M}_R\}.$$

It is also possible to consider the right injective global dimension of R , given by $\sup\{\text{id}(A) \mid A \in \mathcal{M}_R\}$. However, these two values coincide and are sometimes referred to as the right global dimension of R . Of course, by considering left modules one may define the left global dimension of R . It should be noted that the right and left global dimensions of a ring do not always coincide, though in the case that R is left and right noetherian the two values are equal ([37] Corollary 9.23). Throughout this thesis when we refer to the global dimension of a ring, we will mean the right global dimension.

Theorem 1.5.6. The Comparison Theorem.

Suppose that $A, B \in \mathcal{M}_R$ and $f : A \rightarrow B$. Consider the diagram

$$\begin{array}{ccccccc} \cdots & \longrightarrow & P_1 & \xrightarrow{d_1} & P_0 & \xrightarrow{\epsilon} & A \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow f \\ \cdots & \longrightarrow & P'_1 & \xrightarrow{d'_1} & P'_0 & \xrightarrow{\epsilon'} & B \longrightarrow 0 \end{array}$$

where $\cdots \rightarrow P_1 \rightarrow P_0 \rightarrow A \rightarrow 0$ is a projective resolution for A and $\cdots \rightarrow P'_1 \rightarrow P'_0 \rightarrow B \rightarrow 0$ is a projective resolution for B . Then there is a map $\bar{f} = \{\bar{f}_n : P_n \rightarrow P'_n\}$ (represented by the dashed line in the diagram) such that the diagram commutes. □

We say that the map \bar{f} is a **chain map** over f .

Let \mathcal{M}_R denote the category of right R -modules and let $M \in \mathcal{M}_R$. Recall the contravariant functor $T := \text{Hom}_R(-, M)$. Then

$$\begin{aligned} T : \mathcal{M}_R &\longrightarrow \text{sets} \\ A &\mapsto \text{Hom}_R(A, M) \end{aligned}$$

and if $f : A \rightarrow B$, then

$$\begin{aligned} Tf : \text{Hom}_R(B, M) &\longrightarrow \text{Hom}_R(A, M) \\ g &\mapsto gf. \end{aligned}$$

The following definition is an important and well studied example of a right derived functor. The reader is referred to [37] Chapter 6 for more details.

Definition 1.5.7. Consider an R -module A and choose a projective resolution $\dots \rightarrow P_n \xrightarrow{d_n} P_{n-1} \rightarrow \dots \rightarrow P_1 \xrightarrow{d_1} P_0 \rightarrow A \rightarrow 0$ for A . Let $T := \text{Hom}_R(-, M)$ for some $M \in \mathcal{M}_R$ and apply T to the deleted resolution $\dots \rightarrow P_n \xrightarrow{d_n} P_{n-1} \rightarrow \dots \rightarrow P_1 \xrightarrow{d_1} P_0 \rightarrow 0$; that is, form

$$0 \rightarrow \text{Hom}_R(P_0, M) \xrightarrow{Td_1} \text{Hom}_R(P_1, M) \xrightarrow{Td_2} \text{Hom}_R(P_2, M) \rightarrow \dots$$

We define the functor $\text{Ext}_R^n(-, M)$ on \mathcal{M}_R as follows:

$$\text{Ext}_R^n(A, M) := \frac{\ker(Td_{n+1})}{\text{im}(Td_n)},$$

for $A \in \mathcal{M}_R$. We must also define the action of $\text{Ext}_R^n(-, M)$ on a map $f : A \rightarrow B$. By the Comparison Theorem, there is a chain map \bar{f} over f , so we have the commutative diagram,

$$\begin{array}{ccccccc} \dots & \longrightarrow & P_1 & \xrightarrow{d_1} & P_0 & \xrightarrow{\epsilon} & A \longrightarrow 0 \\ & & \downarrow \bar{f}_1 & & \downarrow \bar{f}_0 & & \downarrow f \\ \dots & \longrightarrow & P'_1 & \xrightarrow{d'_1} & P'_0 & \xrightarrow{\epsilon'} & B \longrightarrow 0. \end{array}$$

Then, if $z'_{n+1} \in \ker(Td'_{n+1})$, we define

$$\text{Ext}_R^n(-, M) f : \text{Ext}_R^n(B, M) \longrightarrow \text{Ext}_R^n(A, M)$$

$$\begin{aligned} z'_{n+1} + \text{im}(Td'_n) &\mapsto (T\bar{f}_n) z'_{n+1} + \text{im}(Td_n) \\ &= z'_{n+1} f_n + \text{im}(Td_n). \end{aligned}$$

To simplify the notation we will denote $\text{Ext}_R^n(-, M) f$ by f^* .

Note that $\text{Ext}_R^n(A, M)$ is independent of the choice of projective resolution of A and f^* is independent of the choice of \bar{f} (since any two chain maps over f are homotopic - see [37] for details).

Theorem 1.5.8. *Let $0 \rightarrow M' \xrightarrow{f} M \xrightarrow{g} M'' \rightarrow 0$ be an exact sequence of right R -modules. Then, there is an exact sequence*

$$\begin{aligned} 0 \rightarrow \text{Ext}_R^0(M'', N) \rightarrow \text{Ext}_R^0(M, N) \rightarrow \text{Ext}_R^0(M', N) \rightarrow \dots \\ \rightarrow \text{Ext}_R^n(M'', N) \xrightarrow{g^*} \text{Ext}_R^n(M, N) \xrightarrow{f^*} \text{Ext}_R^n(M', N) \rightarrow \text{Ext}_R^{n+1}(M'', N) \rightarrow \dots \end{aligned}$$

for any right R -module N .

Proof. See [37] Theorem 6.27. □

The following Theorem shows the relationship between the projective dimension of a module M , and $\text{Ext}_R^n(-, M)$.

Theorem 1.5.9. *Let M be a right R -module. The following are equivalent:*

- (i) $\text{pd}(M) \leq n$;
- (ii) $\text{Ext}_R^k(N, M) = 0$ for all right R -modules N and all $k \geq n + 1$;
- (iii) $\text{Ext}_R^{n+1}(N, M) = 0$ for all right R -modules N .

Proof. See [37] Theorem 9.5. □

Let R be a noetherian ring and recall the following definitions.

Definition 1.5.10. The **grade** of a finitely generated (right or left) R -module M is defined by

$$j_R(M) = \inf\{i \mid \text{Ext}_R^i(M, R) \neq 0\} \in \mathbb{N} \cup \{\infty\}.$$

The Auslander condition: The ring R satisfies the Auslander condition if for each finitely generated R -module M , and for all $i \geq 0$ and every R -submodule N of $\text{Ext}_R^i(M, R)$, we have $j_R(N) \geq i$.

A ring of finite injective dimension which satisfies the Auslander condition is called **Auslander Gorenstein**.

A ring of finite global dimension which satisfies the Auslander condition is called **Auslander Regular**.

A k -algebra, where k is a field, is called **Cohen Macaulay** if

$$\mathrm{GKdim}(M) + j_R(M) = \mathrm{GKdim}(R).$$

Theorem 1.5.11. *Let k be a field and let $R = \bigoplus_{n \geq 0} R_n$ be a finitely generated \mathbb{N} -graded k -algebra with $\dim_k R_0 < \infty$. Suppose $x \in R_d$, $d > 0$, is a normal non-zero divisor in R . Then $B = R/Rx$ is Auslander Gorenstein if and only if R is. Furthermore, in this case B is Cohen Macaulay (CM) if and only if R is.*

Proof. See [27] Theorem 5.10. □

Recall that an \mathbb{N} -graded k -algebra $R = \bigoplus_{i \geq 0} R_i$ is said to be **connected** if $R_0 = k$.

Theorem 1.5.12. *Suppose that R is a noetherian ring that is Auslander Regular and Cohen Macaulay. Let $S = R[x; \sigma, \delta]$ be an Ore extension of R .*

(i) *The Ore extension S is Auslander Regular.*

(ii) *Assume that $R = \bigoplus_{i \geq 0} R_i$ is a connected graded k -algebra such that $\sigma(R_i) \subseteq R_i$ for each $i \geq 0$. Then S is Cohen Macaulay.*

Proof. See [28] Lemma. □

Chapter 2

Quantum Grassmannians and Quantum Flag Varieties

Let k be a field. A **Quantum Coordinate Algebra** is a deformation of the coordinate ring of some affine variety which is dependent on a nonzero element q of the field k . One such quantum coordinate algebra is the coordinate ring of $m \times n$ quantum matrices, $\mathcal{O}_q(M_{mn})$ - a quantisation of the coordinate ring of $m \times n$ matrices $\mathcal{O}(M_{m,n}(k))$. This quantum coordinate algebra has been widely studied in recent years (see for example, [12], [18], [21], [28], [32], [36] and many more). In particular, it is a well known result that $\mathcal{O}_q(M_{mn})$ can be constructed as an iterated Ore extension of the field, from which many desirable properties can be obtained. For example, $\mathcal{O}_q(M_{mn})$ is a noetherian domain of Gelfand-Kirillov dimension mn .

In this chapter we study two classes of subalgebras of the coordinate ring of quantum matrices. The first is a deformation of the coordinate ring of grassmannians, and the second is a deformation of the coordinate ring of the flag variety. The chapter begins with the definition of the coordinate ring of quantum matrices, where the reader should note that the relations given by various authors often differ by powers of q . Thus one should take care when using results from works by other authors. We continue by defining the coordinate ring of the quantum Grassmannian and in Section 2.3 a basis for this algebra is constructed, allowing us to calculate its Gelfand-Kirillov dimension. The basis constructed closely follows the construction given in [18] for a basis of the coordinate ring of quantum matrices. In Section 2.4 we define the coordinate ring of the quantum flag variety and proceed to construct a preferred basis using the same methods as before.

2.1 The Coordinate Ring of Quantum Matrices

Throughout this chapter k denotes a field and $0 \neq q \in k$. We have the following definitions.

Definition 2.1.1. The coordinate ring of 2×2 quantum matrices

$$\mathcal{O}_q(M_2) := k \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

is the k -algebra generated by indeterminates a, b, c, d , subject to the following relations:

$$ab = qba; \quad ac = qca; \quad ad - da = (q - q^{-1})bc;$$

$$bc = cb; \quad bd = qbd; \quad cd = qdc.$$

The 2×2 -quantum determinant is given by

$$D = ad - qbc.$$

It is easy to check that D is a central element, while b and c are normal elements of $\mathcal{O}_q(M_2)$.

More generally, let $m, n \in \mathbb{N}$. Then we have the following definition.

Definition 2.1.2. The coordinate ring of $m \times n$ quantum matrices

$$\mathcal{O}_q(M_{mn}) := k \begin{pmatrix} X_{11} & \dots & \dots & X_{1n} \\ X_{21} & \dots & \dots & X_{2n} \\ \vdots & & & \vdots \\ X_{m1} & \dots & \dots & X_{mn} \end{pmatrix},$$

is the k -algebra generated by the mn indeterminates X_{ij} , where $1 \leq i \leq m$ and $1 \leq j \leq n$, subject to the following relations:

$$X_{ij}X_{il} = qX_{il}X_{ij} \quad (j \neq l);$$

$$X_{ij}X_{lj} = qX_{lj}X_{ij} \quad (i \neq l);$$

$$X_{ij}X_{lr} = X_{lr}X_{ij} \quad (i \neq l, r \neq j);$$

$$X_{ij}X_{lr} = X_{lr}X_{ij} + (q - q^{-1})X_{ir}X_{jl} \quad (i \neq l, j \neq r);$$

that is, if we choose 2 elements on the same row or column in the matrix, then they q -commute. Otherwise, the 2 elements will define a 2×2 minor/matrix

whose elements satisfy the relations given above for the 2×2 case.

When $n = m$ we define the **quantum determinant**

$$D = \sum_{\sigma \in S_n} (-q)^{l(\sigma)} X_{1,\sigma(1)} X_{2,\sigma(2)} \cdots X_{n,\sigma(n)},$$

where $l(\sigma)$ denotes the number of inversions in the permutation σ ([36] Lemma 4.1.1).

The quantum determinant D is a central element of $\mathcal{O}_q(M_n)$ ([36] Theorem 4.6.1).

For brevity, we refer to the k -algebra defined above as the algebra of $m \times n$ -**quantum matrices**.

Note that our choice of relations for $\mathcal{O}_q(M_{mn})$ differs slightly from those in [36]; thus whenever we use a result from [36] we must interchange q and q^{-1} .

The algebra of $m \times n$ quantum matrices can be constructed as an iterated Ore extension of the field k ([21] Section 3.1). By studying the commutation relations for $\mathcal{O}_q(M_{mn})$, it is not difficult to see that we can insert the generators X_{ij} into the Ore extension in lexicographic order. In fact we construct the iterated Ore extension

$$k[X_{11}][X_{12}; \sigma_{12}, \delta_{12}] \cdots [X_{1n}; \sigma_{1n}, \delta_{1n}][X_{21}; \sigma_{21}, \delta_{21}] \cdots [X_{mn}; \sigma_{mn}, \delta_{mn}],$$

where

$$\sigma_{st}(X_{ij}) = \begin{cases} q^{-1}X_{ij} & \text{when } i = s, j < t, \\ q^{-1}X_{ij} & \text{when } i < s, j = t, \\ X_{ij} & \text{when } i < s, j \neq t \end{cases}$$

and

$$\delta_{st}(X_{ij}) = \begin{cases} 0 & \text{when } i = s, j < t, \\ 0 & \text{when } i < s, j \geq t, \\ (q^{-1} - q)X_{it}X_{sj} & \text{when } i < s, j < t. \end{cases}$$

It is clear that there is a map from this iterated Ore extension onto $\mathcal{O}_q(M_{mn})$. In fact this map is an isomorphism. The proof of this for 2×2 quantum matrices can be found in [25] IV.4.

The following two results are immediate consequences of this construction.

Lemma 2.1.3. *The algebra of $m \times n$ quantum matrices is a noetherian domain.*

Proof. This result follows from Theorem 1.1.15. □

Lemma 2.1.4.

$$\text{GKdim}(\mathcal{O}_q(M_{mn})) = mn.$$

Proof. This result follows from Proposition 1.3.5. \square

Definition 2.1.5. Consider an $m \times n$ quantum matrix. Let I be a subset of $\{1, \dots, m\}$, and J be a subset of $\{1, \dots, n\}$ with $|I| = l = |J|$. Write I in descending order and J in ascending order. The **quantum minor**, denoted by $[I|J]$, is defined to be the quantum determinant of the $l \times l$ quantum submatrix defined by rows I and columns J .

Example 2.1.6. Consider the algebra of 3×3 -quantum matrices

$$\mathcal{O}_q(M_{33}) = k \begin{pmatrix} X_{11} & X_{12} & X_{13} \\ X_{21} & X_{22} & X_{23} \\ X_{31} & X_{32} & X_{33} \end{pmatrix}.$$

Then $[32|12]$ denotes the quantum minor given by rows 2 and 3, and columns 1 and 2; that is,

$$[32|12] = X_{21}X_{32} - qX_{22}X_{31}.$$

2.2 The Coordinate Ring of Quantum Grassmannians

Recall that in the classical theory the Grassmannian $G(m, n)$ is the set of m dimensional subspaces in \mathbb{C}^n (see [11] pg193 for details). The coordinate ring of the Grassmannian, $\mathcal{O}(G(m, n))$, is the subalgebra of $\mathcal{O}(M_{mn})$ (the coordinate ring of the $m \times n$ matrices) generated by the $m \times m$ minors of the $m \times n$ matrix. This suggests a quantum analogue/deformation $G_q(m, n)$.

Definition 2.2.1. Let $m, n \in \mathbb{N}$ with $n \geq m$. The **coordinate ring of the $m \times n$ quantum Grassmannian**, denoted by $G_q(m, n)$, is defined as the subalgebra of $\mathcal{O}_q(M_{mn})$ generated by the $m \times m$ quantum minors of the $m \times n$ quantum matrix.

For brevity, we refer to the k -algebra defined above as the $m \times n$ **quantum Grassmannian**.

Note. In the quantum Grassmannian $G_q(m, n)$, any minor will involve rows $1, \dots, m$ of the quantum matrix $\mathcal{O}_q(M_{mn})$. Thus to simplify the notation, we may denote a minor by only its columns; that is, the minor given by rows $\{1, \dots, m\}$ and columns J , will be denoted by $[J]$.

Example 2.2.2. $G_q(2, 4)$

$G_q(2, 4)$ is the k -algebra generated by the 2×2 minors of the 2×4 quantum matrices: $[12], [13], [14], [23], [24]$ and $[34]$.

Using the relations for $\mathcal{O}_q(M_{mn})$ and [18] 5.1, we can calculate the following commutation relations:

$$[12][13] = q[13][12], \quad [12][14] = q[14][12], \quad [12][23] = q[23][12],$$

$$[12][24] = q[24][12], \quad [12][34] = q^2[34][12], \quad [13][14] = q[14][13],$$

$$[13][23] = q[23][13], \quad [13][24] = [24][13] + (q - q^{-1})[14][23],$$

$$[13][34] = q[34][13], \quad [14][23] = [23][14], \quad [14][24] = q[24][14],$$

$$[14][34] = q[34][14], \quad [23][24] = q[24][23], \quad [23][34] = q[34][23],$$

$$[24][34] = q[34][24],$$

and the Quantum Plücker relation

$$[12][34] - q[13][24] + q^2[14][23] = 0.$$

Note that since any pair of 2×2 minors will involve at most 4 columns, these relations in fact determine the commutation relations for the more general algebra $G_q(2, n)$.

More generally, given any two minors in $G_q(m, n)$ we can consider their product as a product in $G_q(m, 2m)$ as follows.

A combination of any two minors $[I]$ and $[J]$ will use a maximum of $2m$ columns. Suppose that the columns used between the two minors are $i_1 \leq i_2 \leq \dots \leq i_l$, where $l = 2m - |I \cap J|$. We can identify column i_1 with column 1 of the $m \times 2m$ quantum matrix; column i_2 with column 2 of the $m \times 2m$ quantum matrix; \dots ; column i_l with column l of the $m \times 2m$ quantum matrix. Then the commutation relation in $G_q(m, 2m)$ corresponding to the original product determines the commutation relation for the minors in $G_q(m, n)$ that we started with.

Example 2.2.3. Commutation Relations for $G_q(2, n)$

Let $r < c < s < t \in \{1, \dots, n\}$. Then we can identify the product $[rc][st]$ in

$G_q(2, n)$ with the product [12] [34] in $G_q(2, 4)$. So, from the relations for $G_q(2, 4)$ given above:

$$[rc][st] = q^2 [st][rc].$$

We find the remaining commutation relations for $G_q(2, n)$ in a similar manner. Let $r < c$ and $s < t$. Then

$$[rc][st] = q [st][rc] \quad \text{if } |\{r, c\} \cap \{s, t\}| = 1 \text{ and } r < s \text{ or } c < t;$$

$$[rc][st] = q^2 [st][rc] \quad \text{if } r < c < s < t;$$

$$[rc][st] = [st][rc] \quad \text{if } r < s < t < c;$$

$$[rc][st] = [st][rc] + (q - q^{-1}) [sc][rt] \quad \text{if } r < s < c < t.$$

In [7], Fioresi calculates commutation relations for a general $m \times n$ quantum Grassmannian. However, we must note that in Fioresi's work, $\mathcal{O}_q(M_{mn})$ and $G_q(m, n)$ (and later the quantum Flag variety) are defined over the ring $k[h, h^{-1}]$, where k is an algebraically closed field of characteristic 0, and h is an indeterminate. In particular, we wish to make use of some commutation relations from [7]. Though the restrictions to the field are necessary elsewhere in Fioresi's work, inspection of the proofs reveals that the commutation relations given in [7] Theorem 3.6 remain true over an arbitrary field. Thus we state this result without restricting the field. We require the following total lexicographic ordering on minors:

$$[j_1 j_2 \dots j_m] <_{\text{lex}} [i_1 i_2 \dots i_m] \Leftrightarrow \exists \text{ an index } \alpha \text{ such that } j_l = i_l \text{ for } l < \alpha, \\ \text{but } j_\alpha < i_\alpha.$$

We will denote the version of the $m \times n$ quantum Grassmannian constructed by Fioresi by $\mathcal{G}_h(m, n)$. Note also that the relations in [7] use h where we would use h^{-1} ; thus we should interchange h and h^{-1} .

Proposition 2.2.4. *Let $I, J \subseteq \{1, \dots, 2m\}$ with $|I| = |J| = m$, and $[I] <_{\text{lex}} [J]$. Then in $\mathcal{G}_h(m, n)$,*

$$[I][J] = h^s [J][I] + \sum_{[L] <_{\text{lex}} [I]} \lambda_{[L]} (h - h^{-1})^{i_{[L]}} (-h)^{j_{[L]}} [L][L'],$$

where $\lambda_{[L]} \in k$, $i_{[L]}, j_{[L]} \in \mathbb{N}$, $s = |I \cap J| - m$ and L' is the set $(I \cap J) \cup ((I \cup J) \setminus L)$.

Proof. See [7] Theorem 3.6. □

Using Proposition 2.2.4, one can easily observe that $[I] \in \mathcal{G}_h(m, n)$ is normal modulo the ideal generated by the set $\{[J] : [J] <_{\text{lex}} [I]\}$. We claim that this remains true for our construction of the quantum Grassmannian.

Lemma 2.2.5. *A minor $[I] \in G_q(m, n)$ is normal modulo the ideal generated by the set $\{[J] \mid [J] <_{\text{lex}} [I]\}$.*

Proof. We have a homomorphism

$$\begin{aligned} \theta : \mathcal{G}_h(m, n) &\rightarrow G_q(m, n) \\ h &\mapsto q \\ [I] &\mapsto [I]. \end{aligned}$$

The map θ is certainly onto, but there may be a non-trivial kernel. However, it is clear that $h^s [I] [J] \notin \ker(\theta)$. Therefore

$$\begin{aligned} [I] [J] &= h^s [J] [I] + \sum_{[L] <_{\text{lex}} [I]} \lambda_{[L]} (h - h^{-1})^{i_{[L]}} (-h)^{j_{[L]}} [L] [L'] \\ \Rightarrow \theta([I] [J]) &= \theta \left(h^s [J] [I] + \sum_{[L] <_{\text{lex}} [I]} \lambda_{[L]} (h - h^{-1})^{i_{[L]}} (-h)^{j_{[L]}} [L] [L'] \right) \\ \Rightarrow [I] [J] &= q^s [J] [I] + \theta \left(\sum_{[L] <_{\text{lex}} [I]} \lambda_{[L]} (h - h^{-1})^{i_{[L]}} (-h)^{j_{[L]}} [L] [L'] \right) \\ \Rightarrow [I] [J] &= q^s [J] [I] + \text{"lower terms"}, \end{aligned}$$

where by “lower terms” we mean products of the form $\lambda [L] [L']$ with $\lambda \in k$ and $[L] <_{\text{lex}} [I]$. Therefore $[I]$ is normal modulo the ideal generated by the set $\{[J] \mid [J] <_{\text{lex}} [I]\}$. □

Lemma 2.2.5 shows that the quantum Grassmannian satisfies a property known as “enough normal elements”. Recall that an \mathbb{N} -graded k -algebra A is said to be **connected** if $A = \bigoplus_{i \geq 0} A_i$ with $A_0 = k$.

Definition 2.2.6. If A is a connected \mathbb{N} -graded k -algebra such that every non-simple graded prime factor ring A/P contains a nonzero homogeneous normal element in $\bigoplus_{i \geq 1} (A/P)_i$ then we say that A has **enough normal elements** ([39]).

Suppose that A has a homogeneous normalising sequence $\{x_1, \dots, x_n\}$; that is, x_i is homogeneous and is normal in $A/(x_1, \dots, x_{i-1})$ for all i , and $A/(x_1, \dots, x_n)$ is finite dimensional. Then A has enough normal elements.

We use this property to show that $G_q(m, n)$ is noetherian, and therefore that the quantum Grassmannian is a noetherian domain.

Theorem 2.2.7. *The $m \times n$ quantum Grassmannian $G_q(m, n)$ is a noetherian domain.*

Proof. The quantum Grassmannian $G_q(m, n)$ is generated by the $\binom{n}{m}$ minors of size m in $\mathcal{O}_q(M_{mn})$. Denote these minors by $u_1 <_{\text{lex}} u_2 <_{\text{lex}} \dots <_{\text{lex}} u_{\binom{n}{m}}$ and set $u_0 = 0$. Then by Lemma 2.2.5, $\{u_0, \dots, u_{\binom{n}{m}}\}$ is a normalising subset of $G_q(m, n)$; that is, u_l is normal modulo the ideal generated by $\{u_0, \dots, u_{l-1}\}$, for $l = 0, \dots, \binom{n}{m}$. Let

$$G_l = \frac{G_q(m, n)}{R_l}$$

where R_l is the ideal generated by $u_1, u_2, \dots, u_{\binom{n}{m}-l}$. We will show by induction on l that G_l , for $l = 1, \dots, \binom{n}{m}$, is noetherian.

If $l = 0$, then $G_0 \cong k$ and G_0 is noetherian. Now suppose that G_{l-1} is noetherian. Consider G_l ; from above, $u_{\binom{n}{m}-l+1}$ is normal in G_l . We factor by the ideal generated by $u_{\binom{n}{m}-l+1}$ and note that

$$\frac{G_l}{\langle u_{\binom{n}{m}-l+1} \rangle} \cong G_{l-1}.$$

Thus by [2] Lemma 8.2, G_l is noetherian.

In particular, we have shown that $G_{\binom{n}{m}} = G_q(m, n)$ is noetherian.

Finally, $G_q(m, n)$ is a subalgebra of $\mathcal{O}_q(M_{mn})$ which is a domain, and it follows that $G_q(m, n)$ is a noetherian domain. \square

2.3 A Basis for Quantum Grassmannians

The method used to construct the basis follows exactly the process given in [18] (Section I). In this section we rewrite the relevant results from [18] in terms of quantum Grassmannians.

Let $m, n \in \mathbb{N}$ with $n \geq m$. We define a partial ordering on certain m -element subsets of $\{1, \dots, n\}$.

Definition 2.3.1. Let $A, B \subseteq \{1, \dots, n\}$ with $|A| = m = |B|$. We define the partial ordering, denoted \leq_* . Write A and B in ascending order:

$$A = \{a_1 < a_2 < \dots < a_m\} \quad \text{and} \quad B = \{b_1 < b_2 < \dots < b_m\}.$$

Define $A \leq_* B$ to mean that $a_i \leq b_i$ for $i = 1, \dots, m$.

This naturally defines a partial ordering on the generators of $G_q(m, n)$.

Definition 2.3.2. Let $[I]$ and $[J]$ belong to the generating set of $G_q(m, n)$. Then we write that $[I] \leq_c [J]$ if $I \leq_* J$.

For example, Figure 2.1 shows the ordering on generators of $G_q(3, 6)$.

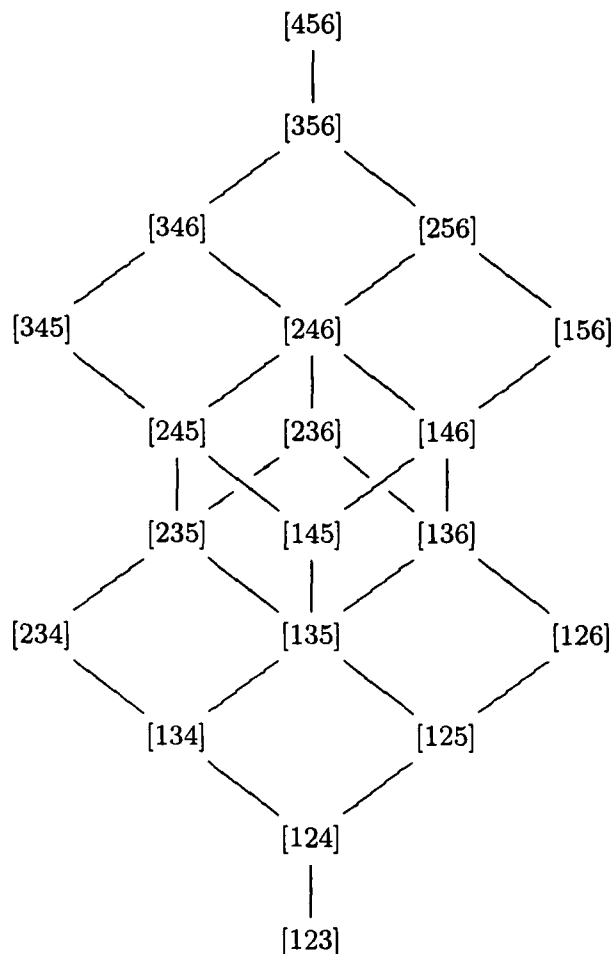


Figure 2.1: The partial ordering \leq_c on $G_q(3, 6)$

The basis of $G_q(m, n)$ is to be constructed and indexed by certain allowable tableaux. Recall the following definition.

Definition 2.3.3. A **Young Diagram** ([10] pg1) is a collection of boxes or cells arranged in left-justified rows with a weakly decreasing number of boxes in each row. A **tableau** is a filling of a Young Diagram; that is, a tableau is any way of putting a positive integer in each cell. A **Young Tableau** ([10] pg2) is a filling of a Young diagram which is:

- (a) strictly increasing along rows; (b) weakly increasing down columns.

Note that the above definition is actually the transpose of the definition appearing in [10].

We consider tableaux (T) , with entries from $\{1, \dots, n\}$. **Allowable tableaux** are of the form:

- (a) each row in (T) has m entries; (b) (T) has strictly increasing rows.

Rows of (T) can be identified with m -element subsets of $\{1, \dots, n\}$ listed in ascending order. Hence allowable tableaux can be labeled (as in [18]) in the form

$$(T) = \begin{pmatrix} J_1 \\ J_2 \\ \vdots \\ J_s \end{pmatrix}.$$

Definition 2.3.4. An allowable tableau (T) is **preferred** if and only if $J_1 \leq_* J_2 \leq_* \dots \leq_* J_s$.

Note that preferred tableaux are in fact Young tableaux.

In order to use induction arguments on allowable tableaux, we define the following ordering:

if

$$(T) = \begin{pmatrix} J_1 \\ J_2 \\ \vdots \\ J_t \end{pmatrix}, \quad (S) = \begin{pmatrix} L_1 \\ L_2 \\ \vdots \\ L_s \end{pmatrix},$$

then $(T) \prec (S)$ if $t > s$, or if $s = t$ and

$$\{J_1, \dots, J_t\} <_{\text{lex}} \{L_1, \dots, L_s\};$$

that is, there exists an index i such that $J_\alpha = L_\alpha$ for $\alpha < i$, but $J_i <_* L_i$.

Any allowable tableau determines a product of quantum minors in the quantum grassmannian as follows.

Definition 2.3.5. For any (allowable) tableau

$$(T) = \begin{pmatrix} J_1 \\ J_2 \\ \vdots \\ J_s \end{pmatrix},$$

define $[T] = [J_1][J_2] \dots [J_s]$.

Definition 2.3.6. The **content** of a tableau (T) is the multiset $\{1^{t_1}, 2^{t_2}, \dots, n^{t_n}\}$, where t_i is the number of times i appears in T .

We will use the content of a tableau to define a natural \mathbb{Z}^n -grading on the $m \times n$ quantum Grassmannian.

Consider a product of minors $[T]$ in $G_q(m, n)$. If the tableau (T) has content $\{1^{t_1}, 2^{t_2}, \dots, n^{t_n}\}$, then we assign the degree (t_1, t_2, \dots, t_n) to $[T]$; that is, $[T]$ is homogeneous of degree (t_1, t_2, \dots, t_n) . One can easily check that this is a valid \mathbb{Z}^n -grading on $G_q(m, n)$, where the degree of a product is dependent on the number of times each column of the $m \times n$ quantum matrix appears in it.

Theorem 2.3.7. *Generalised Quantum Plücker Relations for Quantum Grassmannians*

Let $I = \{1, \dots, n\}$, $J_1, J_2, K \subseteq \{1, 2, \dots, 2n\}$ and $|K| = 2n - |J_1| - |J_2| > n$. Then

$$\sum_{K' \sqcup K'' = K} (-q)^{\ell(J_1; K') + \ell(K'; K'') + \ell(K''; J_2)} [I|J_1 \sqcup K'] [I|K'' \sqcup J_2] = 0,$$

where $\ell(I; J) = |\{(i, j) \in I \times J : i > j\}|$.

Proof. This follows from the Exchange Formulae given in [18] B2(a). Notice that since $|K| > n$, there is no contribution here from the right hand side of the formula. \square

Lemma 2.3.8. *Let (T) be a tableau with content γ and suppose that (T) is not preferred. Then*

- (a) (T) is not minimal with respect to \prec among tableaux with content γ ;
- (b) $[T]$ can be expressed as a linear combination of products $[S]$, where each (S) is a tableau with content γ such that $(S) \prec (T)$.

Proof.(cf. [18] Lemma 1.7.)

Let δ denote the degree of $[T]$. Since (T) is not preferred it must have at least two rows. Write

$$(T) = \begin{pmatrix} J_1 \\ J_2 \\ \vdots \\ J_s \end{pmatrix}.$$

Then $J_j \not\leq_* J_{j+1}$ for some j . We may assume that j is minimal with respect to this property so that $J_1 \leq_* J_2 \leq_* \dots \leq_* J_j$. Let

$$\begin{aligned} J_j &= \{a_1 < a_2 < \dots < a_m\}, \\ J_{j+1} &= \{b_1 < b_2 < \dots < b_m\}. \end{aligned}$$

Then $a_i > b_i$ for some $i \leq n$. Assume i is minimal. So $a_1 \leq b_1, \dots, a_{i-1} \leq b_{i-1}$. Set

$$\begin{aligned} A_1 &= \{a_1 < a_2 < \dots < a_{i-1}\}, \\ A_2 &= \{b_{i+1} < \dots < b_m\}, \\ K &= \{b_1 < \dots < b_i < a_i < \dots < a_m\}. \end{aligned}$$

Since $\{b_1, \dots, b_i\}$ has one more element than A_1 , there must be an index $p \leq i$ such that $b_p \notin A_1$. In addition, $b_p \leq b_i < a_i < \dots < a_m$, and so $b_p \notin J_j$. Similarly, there is an index $q \geq i$ such that $a_q \notin J_{j+1}$, and $b_p \leq b_i < a_i \leq a_q$. Now set

$$\begin{aligned} J'_j &= J_j \cup \{b_p\} \setminus \{a_q\}, \\ J'_{j+1} &= J_{j+1} \cup \{a_q\} \setminus \{b_p\}. \end{aligned}$$

Notice that $J'_j \cup J'_{j+1} = J_j \cup J_{j+1}$, and $J'_j <_* J_j$ because $b_p < a_q$. Let

$$(R) = \begin{pmatrix} J_1 \\ J_2 \\ \vdots \\ J_{j-1} \\ J'_j \\ J'_{j+1} \\ J_{j+2} \\ \vdots \\ J_s \end{pmatrix}.$$

Then (R) is a tableau with the same content as (T) , and since $J'_j \leq_* J_j$, we have $(R) \prec (T)$.

(b) The Generalised Quantum Plucker relations (Theorem 2.3.7) give us a relation of the form

$$\sum_{K=K' \sqcup K''} (-q)^{\ell(A_1; K') + \ell(K'; K'') + \ell(K''; A_2)} [A_1 \sqcup K'] [K'' \sqcup A_2] = 0, \quad (\dagger)$$

with all the terms of the same degree. Note that $[J_j][J_{j+1}]$ occurs in (\dagger) when $K' = \{a_i < \dots < a_m\}$ and $K'' = \{b_1 < \dots < b_i\}$. In any other term on the left K' contains at least one of b_1, \dots, b_i , from which we see that $A_1 \sqcup K' <_* J_j$. So we have the relation

$$\sum_{K=K' \sqcup K''} (-q)^* [J_1] \cdots [J_{j-1}] [A_1 \sqcup K'] [K'' \sqcup A_2] [J_{j+2}] \cdots [J_l] = 0,$$

where $*$ $= \ell(A_1; K') + \ell(K'; K'') + \ell(K''; A_2)$, and all terms are of the same degree as $[T]$.

The term $[T]$ occurs once. All the other terms are of the form $\pm q^\bullet [S]$, where

$$(S) = \begin{pmatrix} J_1 \\ \vdots \\ J_{j-1} \\ A_1 \sqcup K' \\ K'' \sqcup A_2 \\ J_{j+2} \\ \vdots \\ J_s \end{pmatrix}$$

for some $K' \neq \{a_i < \dots < a_n\}$. Now, $A_1 \sqcup K' <_* J_j$ and thus $(S) \prec (T)$. Therefore we have the result. \square

Theorem 2.3.9. *Let $\delta = (c_1, \dots, c_n)$, let V be the homogeneous component of $G_q(m, n)$ with degree δ , and set $\gamma = (1^{c_1} 2^{c_2} \dots n^{c_n})$. The products $[T]$, as (T) runs over all preferred tableau with content γ , form a basis for V .*

Proof.

We use induction on \prec to show that the products $[T]$, as (T) runs over all preferred tableau with content γ , span V .

Suppose that (U) is minimal with respect to \prec in V . Then by the previous result the tableau (U) is preferred.

Suppose that for any allowable tableau $(S') \prec (S)$ with content γ , we can rewrite the product $[S']$ as a linear combination of products $[S_i] \in V$, where each (S_i) is preferred. Consider the product $[S]$. By Lemma 2.3.8, $[S]$ can be expressed as a linear combination of products $[U] \in V$, with $(U) \prec (S)$. By the induction hypothesis, we can rewrite each of the products $[U]$ as a linear combination of preferred products in V . Therefore $[S]$ can be written as a linear combination of preferred products. Hence the products $[T]$, as (T) runs over all preferred tableaux with content γ , span V .

Recall that $G_q(m, n)$ is a subalgebra of $\mathcal{O}_q(M_{mn})$ and notice that the products $[T]$, as (T) runs over all preferred tableaux of content γ , form a subset of the basis of $\mathcal{O}_q(M_{mn})$ constructed in [18]. Therefore, they are linearly independent and we have the result. \square

Corollary 2.3.10. *The products $[T]$, as (T) runs over all preferred tableaux, form a basis for $G_q(m, n)$.* \square

This basis can be used to calculate the Gelfand-Kirillov dimension of the $m \times n$ quantum Grassmannian.

Consider the diagram of the partial ordering \leq_c on the minors of $G_q(m, n)$. A **saturated path** between two minors $a <_c b$ will be an ‘upwards path’ $a = a_1 <_c a_2 <_c \dots <_c a_l = b$ of minors such that no additional terms can be added; that is, for any index i there is no minor d such that $a_i <_c d <_c a_{i+1}$. The **length** of such a saturated path is defined to be l .

Example 2.3.11. An example of a saturated path between the minors $[134]$ and $[256]$ in $G_q(3, 6)$ is

$$[134] <_c [234] <_c [235] <_c [236] <_c [246] <_c [256].$$

The length of this saturated path is 6.

A **maximal path** is a saturated path between the two minors $[1 \dots m]$ and $[n - m + 1 \dots n]$. We claim that any maximal path will have length

$$(n - m + 1 + n - m + 2 + \dots + n) - (1 + 2 + \dots + n) + 1 = m(n - m) + 1.$$

First we describe one example of a maximal path and show that it has the desired length. We will need the following definition from [7].

Definition 2.3.12. Let m, n be natural numbers and let $I = \{i_1 < \dots < i_m\} \subseteq \{1, \dots, n\}$. If $I \neq \{1, \dots, m\}$, let p be the least integer such that $i_p - i_{p-1} > 1$ ($i_0 = 0$). Let σ be the elementary transposition $\sigma = (i_p \ i_p - 1)$ and call this transposition the **standard transposition** of I . We define the **standard tower** of $[I]$ to be the sequence

$$[I] = [I_N], [I_{N-1}], \dots, [I_1], [I_0] = [1 \dots m] \quad \text{where} \quad I_j = \sigma_{j+1}(I_{j+1})$$

and σ_j is the standard transposition of I_j .

Note: If $[I] = [I_N], [I_{N-1}], \dots, [I_1], [I_0] = [1 \dots m]$ is the standard tower for I , we have

$$[I] = [I_N] >_c [I_{N-1}] >_c \dots >_c [I_1] >_c [I_0] = [1 \dots m].$$

Example 2.3.13. Consider the minor $[34]$ in the 2×5 quantum Grassmannian $G_q(2, 5)$. The standard tower of $[34]$ is

$$[34] >_c [24] >_c [14] >_c [13] >_c [12].$$

Let $[n + m - 1 \dots n] = [I_N], [I_{N-1}], \dots, [I_1], [I_0] = [1 \dots m]$ be the standard tower for $[n - m + 1 \dots n]$ in $G_q(m, n)$. Then it is clear that $[I_0] = [1 \dots m] <_c$

$[I_1] <_c \dots <_c [I_N] = [n - m + 1 \dots n]$ is a maximal path in $G_q(m, n)$. We will call this the **standard maximal path** in $G_q(m, n)$. In the standard tower for $[n - m + 1 \dots n]$ each entry must be decreased by 1 exactly $n - m$ times and there are m entries. Therefore the length of the standard maximal path is $m(n - m) + 1$.

Consider a maximal path

$$[1 \dots m] = a_1 <_c a_2 <_c \dots <_c a_l = [n - m + 1 \dots n]$$

in $G_q(m, n)$. Let $a_i = [\alpha_1 \dots \alpha_m] <_c a_{i+1} = [\beta_1 \dots \beta_m]$ for some $1 \leq i \leq l - 1$. Clearly $\sum \alpha_i < \sum \beta_i$. We claim that $\sum \alpha_i + 1 = \sum \beta_i$. Suppose that $\sum \alpha_i + 2 \leq \sum \beta_i$. Then either there exists an index j with $\alpha_j + 1 < \beta_j$ or there are two indices $j < k$ such that $\alpha_j < \beta_j$ and $\alpha_k < \beta_k$. Suppose we have an index j with $\alpha_j + 1 < \beta_j$. Then

$$[\alpha_1 \dots \alpha_j \dots \alpha_m] <_c [\alpha_1 \dots \alpha_j + 1 \dots \alpha_m] <_c [\beta_1 \dots \beta_j \dots \beta_m],$$

which is a contradiction. Now suppose there are two indices $j < k$ such that $\alpha_j < \beta_j$ and $\alpha_k < \beta_k$. Then

$$[\alpha_1 \dots \alpha_j \dots \alpha_k \dots \alpha_m] <_c [\alpha_1 \dots \alpha_j + 1 \dots \alpha_k \dots \alpha_m] <_c [\beta_1 \dots \beta_j \dots \beta_k \dots \beta_m],$$

and again we have a contradiction. Therefore $\sum \alpha_i + 1 = \sum \beta_i$. Thus in any maximal path we may only increase one entry of a minor by exactly one at any point in the path. To get from $[1 \dots m]$ to $[n - m + 1 \dots n]$ in this manner will take

$$(n - m + 1 - 1) + (n - m + 2 - 2) + \dots + (n - m + m - m) = m(n - m)$$

steps. Therefore the length of any maximal path in $G_q(m, n)$ is $m(n - m) + 1$.

We are now in a position to calculate the Gelfand-Kirillov dimension of $G_q(m, n)$.

Proposition 2.3.14. *Let $G = G_q(m, n)$ and let α be the length of a maximal path in G . Then*

$$\text{GKdim}(G_q(m, n)) = \alpha = m(n - m) + 1.$$

Proof. Let V be the k -subspace of G spanned by the $m \times m$ minors which generate G . Then

$$d_V(n) = \dim_k \left(\sum_{i=0}^n V^i \right) \quad \text{and} \quad \text{GKdim}(G) = \overline{\lim} \log_n d_V(n).$$

Let $a_1, a_2, \dots, a_\alpha$ be a maximal path in G . Then

$$a_1^{s_1} a_2^{s_2} \dots a_\alpha^{s_\alpha} \in V^{n+1} \text{ where } \sum_{i=1}^{\alpha} s_i = n + 1.$$

The set $\{a_1^{s_1} a_2^{s_2} \dots a_\alpha^{s_\alpha} \mid \sum s_i = n + 1\}$ is linearly independent. Therefore

$$\begin{aligned} \dim_k(V^{n+1}) &\geq |\{a_1^{s_1} a_2^{s_2} \dots a_\alpha^{s_\alpha} \mid \sum s_i = n + 1\}| \\ &= \binom{n + \alpha}{\alpha - 1} \end{aligned}$$

which is a polynomial in n of degree $\alpha - 1$. So by Lemma 1.3.2 (i), we can rewrite this as

$$\binom{n + \alpha}{\alpha - 1} = z_\alpha \binom{n}{\alpha - 1} + \dots + z_2 \binom{n}{1} + z_1$$

for some $z_i \in \mathbb{Q}$ with $z_\alpha \neq 0$. Then

$$d_V(n + 1) - d_V(n) \geq z_\alpha \binom{n}{\alpha - 1} + \dots + z_2 \binom{n}{1} + z_1.$$

So by Lemma 1.3.2 (ii)

$$d_V(n) \geq z_\alpha \binom{n}{\alpha} + \dots + z_1 \binom{n}{1} + z_0,$$

which is a polynomial in n of degree α . Therefore

$$\text{GKdim}(G) = \overline{\lim} \log_n d_V(n) \geq \alpha.$$

Let $a_{i_1} \dots a_{i_n} \in V^n$. Note that in Lemma 2.3.8 when we rewrite a product of length n in terms of preferred products, the length of these preferred products is also n . Then since preferred products form a basis for V :

$$a_{i_1} \dots a_{i_n} = \sum (\text{preferred products of length } n).$$

There are finitely many maximal paths in $G_q(m, n)$. Suppose there are c such paths and index them $1, \dots, c$. Let $a_1 <_c a_2 <_c \dots <_c a_\alpha$ be the i th maximal path and let W_i^n denote the subspace generated by monomials

$$a_1^{s_1} \dots a_\alpha^{s_\alpha} \text{ such that } \sum s_j = n.$$

Then $V^n \subseteq \sum_{i=1}^c W_i^n$. Consider $\dim(W_i^n)$. The products $a_1^{s_1} \dots a_\alpha^{s_\alpha}$ such that $\sum s_j = n$ are linearly independent. Therefore

$$\begin{aligned} \dim(W_i^n) &= |\{a_1^{s_1} \dots a_\alpha^{s_\alpha} \mid \sum s_i = n\}| \\ &= |\{(s_1, \dots, s_\alpha) \in \mathbb{N}^\alpha \mid \sum s_i = n\}|. \end{aligned}$$

Therefore $\dim(W_i^n) = \dim(W_j^n)$ for all $i, j \in \{1, \dots, c\}$. Thus

$$\dim(V^n) \leq \dim\left(\sum_{i=1}^c W_i^n\right) \leq c \dim(W_1^n)$$

and $d_V(n) \leq c d_{W_1}(n)$. Therefore

$$\begin{aligned} \text{GKdim}(G) &= \overline{\lim} \log_n d_V(n) \\ &\leq \overline{\lim} \log_n c d_{W_1}(n) \\ &= \lim_{n \rightarrow \infty} \log_n c + \overline{\lim} \log_n d_W(n) \\ &= \lim_{n \rightarrow \infty} \log_n c + \alpha \\ &= \alpha. \end{aligned}$$

Therefore $\text{GKdim}(G) = \alpha = m(n - m) + 1$. \square

Example 2.3.15. Figure 2.2 shows the partial ordering \leq_c on $G_q(2, 5)$ and the dotted line shows the standard maximal path.

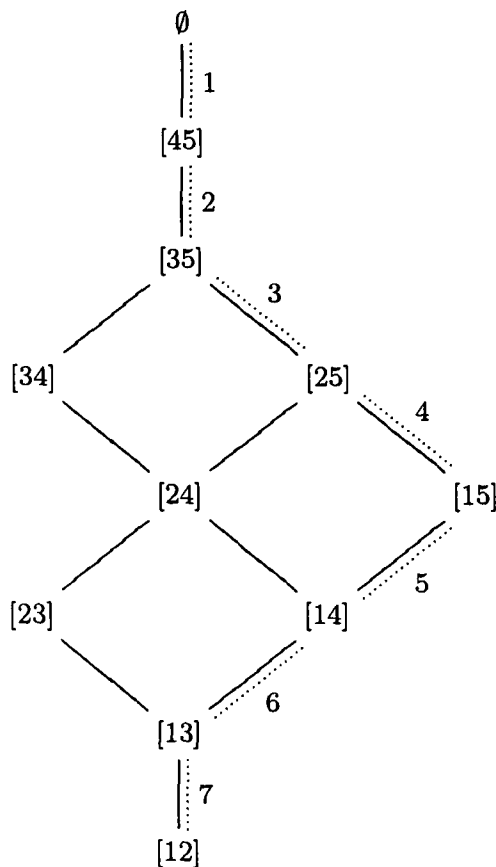


Figure 2.2: The partial ordering \leq_c on $G_q(2, 5)$

Therefore $\text{GKdim}(G_q(2, 5)) = 7$.

2.4 Quantum Flag Varieties

In the classical theory a **flag** is a strictly increasing sequence of subspaces in \mathbb{C}^n . For a fixed set $\{i_1, \dots, i_s\} \subseteq \mathbb{N}$, such that $1 \leq i_1 < \dots < i_s \leq n$, the **flag variety**

$F = F(i_1, \dots, i_s; n)$ is the set of all flags

$$0 < V_{i_1} < V_{i_2} < \dots V_{i_s} \leq V = \mathbb{C}^n$$

such that $\dim V_{i_r} = i_r$, for $1 \leq r \leq s$. The **coordinate ring of the flag variety** $\mathcal{O}(F)$ is generated by the $i_r \times i_r$ minors of the $i_r \times n$ matrices ($1 \leq r \leq s$) and is therefore a subalgebra of the coordinate ring of $i_s \times n$ matrices. This suggests a quantum analogue/deformation.

Definition 2.4.1. Let $1 \leq i_1 < \dots < i_s \leq n$. The **coordinate ring of the quantum flag variety**, $F_q(i_1, \dots, i_s; n)$ is the k -subalgebra of the $i_s \times n$ quantum matrices generated by the $i_r \times i_r$ quantum minors formed using rows $1, 2, \dots, i_r$ of the $i_r \times n$ quantum matrices.

For brevity, we will refer to $F_q(i_1, \dots, i_s; n)$ as the **quantum flag variety**. Note that in the quantum flag variety $F_q(i_1, \dots, i_s; n)$, any $i_r \times i_r$ minor will involve rows $1, \dots, i_r$ of the quantum matrix $\mathcal{O}_q(M_{i_s, n})$. Thus we will denote a minor by only its columns; that is, the minor given by rows $1, \dots, i_r$ and columns J will be denoted by $[J]$.

The definition is clarified when we consider some examples.

Example 2.4.2. The quantum flag variety $F_q(1, 2; 5)$ is the k -subalgebra of

$$\mathcal{O}_q(M_{2,5}) = \begin{pmatrix} X_{11} & X_{12} & X_{13} & X_{14} & X_{15} \\ X_{21} & X_{22} & X_{23} & X_{24} & X_{25} \end{pmatrix}$$

generated by the 1×1 minors

$$X_{11}, X_{12}, X_{13}, X_{14}, X_{15}$$

and the 2×2 minors

$$[12], [13], [14], [15], [23], [24], [25], [34], [35], [45].$$

Using the relations from [18] and the commutation relations given in Example 2.2.3 for $G_q(2, n)$, it is possible to calculate the commutation relations for $F_q(1, 2; 5)$.

Example 2.4.3. The $m \times n$ quantum Grassmannian is an example of a quantum flag variety:

$$G_q(m, n) = F_q(m; n).$$

Of course, the results regarding quantum Grassmannians in the previous sections can be obtained as special cases of those appearing here and in Section 2.5. However, since the quantum Grassmannian is the focus for much of Chapters 3 and 4, it seems reasonable to present the results separately. In several of the proofs of results for the quantum flag variety the reader is referred to the proof of the corresponding result for quantum Grassmannians.

Fioresi [8] has also done some work on quantum flag varieties. However, as with results on quantum Grassmannians, one should be aware of the differences in the definitions given in [8] and those given here. We denote the version of the quantum flag variety constructed by Fioresi by $\mathcal{F}_h(i_1, \dots, i_s; n)$, which is defined over the ring $k[h, h^{-1}]$, where k is an algebraically closed field of characteristic 0 and h is an indeterminate. In particular, Fioresi has calculated commutation relations for $\mathcal{F}_h(i_1, \dots, i_s; n)$ (for which the restrictions to the field are unnecessary). We have a total lexicographic ordering on minors: $[r_1 \dots r_{i_\alpha}] <_{\text{lex}} [s_1 \dots s_{i_\beta}]$ if and only if there exists an index m such that $r_l = s_l$ for $l < m$ but $r_m < s_m$ or $i_\alpha < i_\beta$ and $r_m = s_m$ for $m = 1, \dots, i_\alpha$.

Proposition 2.4.4. *Let $I, J \subseteq \{1, \dots, n\}$ with $|I| = i_\alpha$ and $|J| = i_\beta$ for some $1 \leq \alpha, \beta \leq s$. Let $[I] <_{\text{lex}} [J]$. Then in $\mathcal{F}_h(i_1, \dots, i_s; n)$,*

$$[I][J] = q^{\min\{i_\alpha, i_\beta\}} [J][I] + \sum_{[L] <_{\text{lex}} [I]} \lambda_{[L]} (h - h^{-1})^{l_{[L]}} (-h)^{m_{[L]}} [L][L'],$$

where $\lambda_{[L]} \in k$, $l_{[L]}, m_{[L]} \in \mathbb{N}$ and L' is the set $(I \cap J) \cup ((I \cup J) \setminus L)$. In fact the set $L \cup L'$ (with repetitions) is equal to the set $I \cup J$ (with repetitions) and $|L| = |I|$ and $|L'| = |J|$.

Proof. See [8] Theorem 3.8. □

From Proposition 2.4.4 one easily observes that a minor $[I] \in \mathcal{F}_h(i_1, \dots, i_s; n)$ is normal modulo the ideal generated by those minors beneath $[I]$ in the lexicographic ordering. We claim that this remains true in $F_q(i_1, \dots, i_s; n)$.

Lemma 2.4.5. *A minor $[I] \in F_q(i_1, \dots, i_s; n)$ is normal modulo the ideal generated by the set $\{[J] \mid [J] <_{\text{lex}} [I]\}$.*

Proof. The reader is referred to the proof of Lemma 2.2.5, the corresponding result for quantum Grassmannians. □

Proposition 2.4.6. *The quantum flag variety $F_q(i_1, \dots, i_s; n)$ is a noetherian domain.*

Proof. Let $F := F_q(i_1, \dots, i_s; n)$. Then F is generated by $\sum_{t=i}^s \binom{n}{i_t} = \alpha$ elements on which we have a total lexicographic ordering. Denote these elements

$$0 = u_0 <_{\text{lex}} u_1 <_{\text{lex}} \dots <_{\text{lex}} u_\alpha.$$

Let $F_l = F / \langle u_1, \dots, u_{\alpha-l} \rangle$ for $l = 0, 1, \dots, \alpha$. We claim that F_l is noetherian. We use induction on l . If $l = 0$, then $F_l = F_0 = F / \langle u_1, \dots, u_\alpha \rangle \cong k$ and F_0 is noetherian. Now suppose that F_l is noetherian and consider F_{l+1} . Then $F_{l+1} = F / \langle u_1, \dots, u_{\alpha-l-1} \rangle$ and $u_{\alpha-l}$ is normal in F_{l+1} . Factor F_{l+1} by $u_{\alpha-l}$:

$$\frac{F_{l+1}}{\langle u_{\alpha-l} \rangle} = \frac{F}{\langle u_1, \dots, u_{\alpha-l-1}, u_{\alpha-l} \rangle} = F_l.$$

Therefore, by the induction hypothesis, $F_{l+1} / \langle u_{\alpha-l} \rangle$ is noetherian. Thus by [2] Lemma 8.2, F_{l+1} is noetherian, and this holds for $l = 0, \dots, \alpha$. In particular we have that $F_\alpha = F$ is noetherian. \square

2.5 A Basis for the Quantum Flag Variety

The basis constructed here is a generalisation of the basis constructed in Section 2.3 for quantum Grassmannians and so follows the process given in [18]. First we define a partial ordering on certain subsets of $\{1, \dots, n\}$.

Definition 2.5.1. Let $1 \leq \alpha, \beta \leq s$ and let $A, B \subseteq \{1, \dots, n\}$ such that $|A| = i_\alpha$ and $|B| = i_\beta$. Write A and B in ascending order:

$$A = \{a_1 < a_2 < \dots < a_{i_\alpha}\} \quad \text{and} \quad B = \{b_1 < b_2 < \dots < b_{i_\beta}\}.$$

Then $A \leq_* B$ if and only if $i_\alpha \geq i_\beta$ and $a_i \leq b_i$ for $i = 1, \dots, i_\beta$.

Let $F := F_q(i_1, \dots, i_s; n)$. Then the partial ordering \leq_* naturally defines a partial ordering on the generators of F .

Definition 2.5.2. Let $[I], [J]$ belong to the generating set of F and define $[I] \leq_c [J]$ to mean that $I \leq_* J$.

Example 2.5.3. Figure 2.3 shows the partial ordering \leq_c on the generators of $F_q(1, 2; 5)$.

As we have seen in the case of quantum Grassmannians, the basis will be indexed by certain tableaux. Consider tableaux with entries from 1 to n and no repetitions in any row. A tableau (T) is said to be **allowable** if



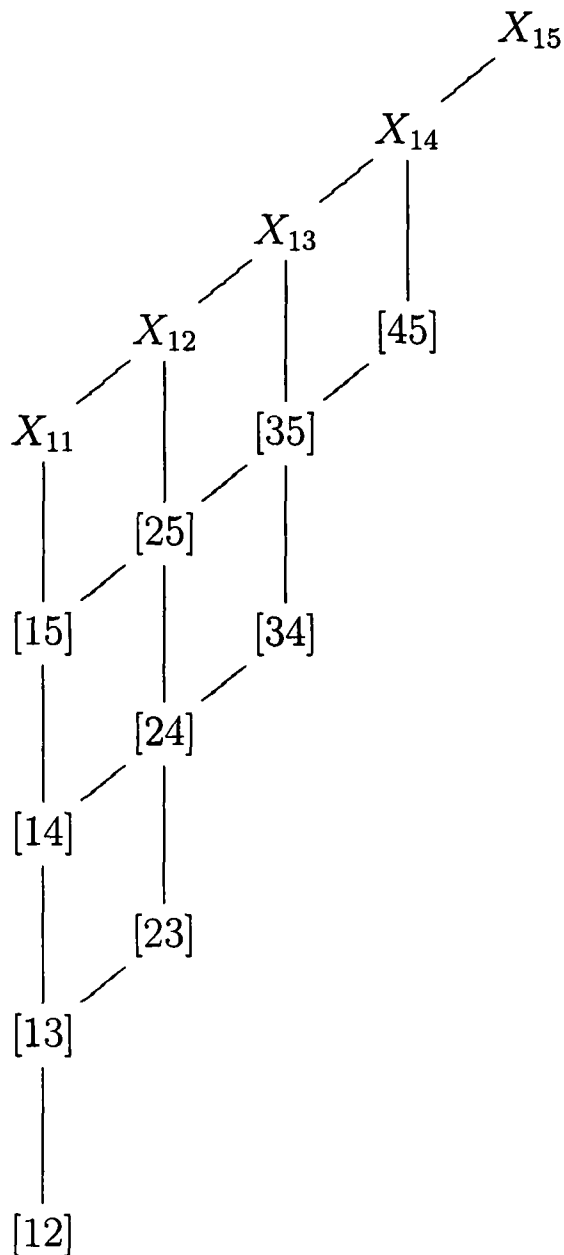


Figure 2.3: The partial ordering \leq_c on $F_q(1, 2; 5)$

- (i) each row in (T) has i_α entries for some $1 \leq \alpha \leq s$;
- (ii) the rows of (T) are strictly increasing.

A tableau (T) is **preferred** if

- (i) it is allowable;
- (ii) The columns of (T) are non-decreasing.

Allowable tableaux can be written $(T) = \begin{pmatrix} J_1 \\ \vdots \\ J_t \end{pmatrix}$ and (T) is preferred if and only if $J_1 \leq_* \dots \leq_* J_t$. For induction purposes we require an ordering on allowable tableaux. Recall that given a tableau $(T) = \begin{pmatrix} J_1 \\ J_2 \\ \vdots \\ J_t \end{pmatrix}$, the **shape** of (T) is defined

by

$$\text{shape}(T) := (|J_1|, |J_2|, \dots, |J_t|) \in \mathbb{N}^t.$$

Definition 2.5.4. Let $(T) = \begin{pmatrix} J_1 \\ J_2 \\ \vdots \\ J_t \end{pmatrix}$ and $(S) = \begin{pmatrix} L_1 \\ L_2 \\ \vdots \\ L_l \end{pmatrix}$. Then $(T) \prec (S)$ if and only if

(i) the shape of (T) is larger than the shape of (S) relative to the lexicographic ordering on shapes, or

(ii) (T) and (S) have the same shape and

$$(L_1, \dots, L_l) < (J_1, \dots, J_t);$$

that is, there exists an index i such that $L_\alpha = J_\alpha$ for $\alpha < i$ but $L_i <_* J_i$.

Any allowable tableau $(T) = \begin{pmatrix} J_1 \\ J_2 \\ \vdots \\ J_t \end{pmatrix}$ defines a product in F . Define $[T] := [J_1][J_2] \dots [J_t]$. Such a product is said to be **preferred** if the tableau (T) is preferred.

Definition 2.5.5. The **content** of a tableau (T) is defined to be the multiset $\{1^{m_1}, 2^{m_2}, \dots, n^{m_n}\}$, where m_i is the number of times i appears in (T) .

Shortly, we will see that by using the content of allowable tableaux we can define a \mathbb{Z}^n -grading on F . We have seen that any allowable tableau determines a product in F . The aim now is to show that any product in F can be rewritten as a linear combination of products which can be represented by allowable tableaux. First we will show that given any two minors in F , their product can be expressed as a linear combination of products represented by allowable tableaux. The proof of this relies on using the commutation relations from [8] for our version of the quantum Flag variety. We state the relevant result in our terms (Proposition 2.4.4 restated).

Lemma 2.5.6. Let $I, J \subseteq \{1, \dots, n\}$ with $|I| = i_\alpha$, $|J| = i_\beta$ for some $1 \leq \alpha, \beta \leq s$. Let $[I] <_{\text{lex}} [J]$. Then

$$[I][J] = q^{\min\{i_\alpha, i_\beta\}} [J][I] + N,$$

where N is a linear combination of products $[L][L']$ such that

(i) $[L] <_{\text{lex}} [I]$;

(ii) $L' = (I \cap J) \cup ((I \cup J) \setminus L)$ and in fact the set $L \cup L'$ (with repetitions) is equal to the set $I \cup J$ (with repetitions);

(iii) $|L| = |I|$ and $|L'| = |J|$.

Proof. This result follows from Proposition 2.4.4 (and the existence of a homomorphism similar to that in the proof of Lemma 2.2.5). \square

Lemma 2.5.7. *Given any two minors $[J]$ and $[C]$ in F , if $\binom{J}{C}$ is not a tableau (so $|J| < |C|$) we can rewrite the product $[J][C]$ as a linear combination of products which are represented by allowable tableaux (I) , say. Furthermore, each tableau (I) has the same content as the tableau $\binom{C}{J}$.*

Proof. We use induction on $[J]$ with respect to the lexicographic ordering on minors. The minor minimal with respect to such an ordering is $[1 \dots i_1]$. Suppose that $[J] = [1 \dots i_1]$. We will show that for any minor $[C]$ we can rewrite $[J][C]$ in the required form. We may assume that $[C] \neq [J]$ and therefore

$$[J] <_{\text{lex}} [C].$$

Thus by Lemma 2.5.6

$$[J][C] = q^{i_1} [C][J].$$

The tableau $\binom{C}{1 \dots i_1}$ is clearly allowable and so we are done.

Now suppose that $[1 \dots i_1] <_{\text{lex}} [J]$ and that if $[S] <_{\text{lex}} [J]$, then given any minor $[C]$, we can rewrite the product $[S][C]$ in the required form. If $|J| \geq |C|$, then $\binom{J}{C}$ is an allowable tableau. Suppose $|J| < |C|$; then there are two cases to deal with.

Case 1. $[J] <_{\text{lex}} [C]$

By Lemma 2.5.6

$$[J][C] = q^{\min\{|J|, |C|\}} [C][J] + N,$$

where N is a linear combination of products of the form $[L][L']$ with $|L| = |J|$ and $|L'| = |C|$. Thus $\binom{L}{L'}$ is not an allowable tableau. However $[L] <_{\text{lex}} [J]$, so by the induction hypothesis we can rewrite $[L][L']$ as a linear combination of

products given by allowable tableau (I) which have the same content as $\begin{pmatrix} L' \\ L \end{pmatrix}$, which has the same content as $\begin{pmatrix} C \\ J \end{pmatrix}$. Since $\begin{pmatrix} C \\ J \end{pmatrix}$ is allowable we are done.

Case 2. $[C] <_{\text{lex}} [J]$

By Lemma 2.5.6

$$q^{\min\{|J|, |C|\}} [J] [C] = [C] [J] + M,$$

where M is a linear combination of products of the form $[L] [L']$ where $|L| = |C|$ and $|L'| = |J|$. So the tableau $(I) = \begin{pmatrix} L \\ L' \end{pmatrix}$ is allowable and has the same content as $\begin{pmatrix} C \\ J \end{pmatrix}$. \square

Lemma 2.5.8. *Let $[C_1], [C_2], \dots, [C_l]$ be minors in F and let σ be a permutation on $\{1, \dots, l\}$ such that $|C_{\sigma(1)}| \geq \dots \geq |C_{\sigma(l)}|$. Then the product $[C_1] [C_2] \dots [C_l]$ can be expressed as a linear combination of products $[T]$ which can be represented by allowable tableaux (T) with the same shape and content as the allowable tableau $\begin{pmatrix} C_{\sigma(1)} \\ \vdots \\ C_{\sigma(l)} \end{pmatrix}$.*

Proof. This result follows from Lemma 2.5.7. \square

Thus the content of allowable tableaux allows us to place an \mathbb{Z}^n -grading on F . If a tableau (T) has content $\{1^{m_1}, 2^{m_2}, \dots, n^{m_n}\}$, then we assign the degree (m_1, \dots, m_n) to the product $[T]$; that is, $[T]$ is homogeneous of degree (m_1, \dots, m_n) . So the degree of a product is dependent on the number of times each column of the $i_s \times n$ matrix appears in it.

Lemma 2.5.9. *Generalised Quantum Plücker Relations for Quantum Flag Varieties*

Let $F = F_q(i_1, \dots, i_s; n)$ and let $I_1, I_2, J_1, J_2, K \subseteq \{1, \dots, n\}$ such that $|I_\nu| = i_{\alpha_\nu}$, $|J_\nu| = i_{\beta_\nu}$ for some $\alpha_\nu, \beta_\nu \in \{i_1, \dots, i_s\}$ and $|K| = |I_1| + |I_2| - |J_1| - |J_2| > \max\{i_{\alpha_\nu}\}$. Then

$$\sum_{K' \sqcup K'' = K} (-q)^\bullet [I_1 | J_1 \sqcup K'] [I_2 | K'' \sqcup J_2] = 0$$

where $\bullet = \ell(J_1 : K') + \ell(J_1 : J_2) + \ell(J_2 : K'')$.

Proof. The result follows from [18] Proposition B2(a), noting that the right hand side is zero since, $|K| > \max\{i_{\alpha_\nu}\}$. \square

Lemma 2.5.10. *Let (T) be an allowable tableau with content γ and suppose that (T) is not preferred. Then*

(i) *(T) is not minimal with respect to \prec among tableaux of content γ ;*

(ii) *$[T]$ can be expressed as a linear combination of products $[S]$, where each (S) is an allowable tableau with content γ such that $(S) \prec (T)$.*

Proof. Let $(T) = \begin{pmatrix} J_1 \\ \vdots \\ J_t \end{pmatrix}$. Since (T) is not preferred $t \geq 2$ and $J_j \not\prec_* J_{j+1}$ for some j . Assume that j is minimal with respect to this property and let

$$J_j = \{a_1 < a_2 < \dots < a_\alpha\},$$

$$J_{j+1} = \{b_1 < b_2 < \dots < b_\beta\},$$

where $\alpha = i_l$ and $\beta = i_m$ for some $1 \leq m < l \leq s$. Then $a_i > b_i$ for some $i \leq \beta$. Assume that i is minimal with respect to this property and proceed as in the proof of Lemma 2.3.8 with

$$A_1 = \{a_1 < a_2 < \dots < a_{i-1}\},$$

$$A_2 = \{b_{i+1} < \dots < b_\beta\},$$

$$K = \{b_1 < \dots < b_i < a_i < \dots < a_\alpha\}.$$

(ii) From the generalised Quantum Plücker relation we obtain

$$\sum_{K=K' \sqcup K''} (-q)^{\ell(A_1; K') + \ell(K'; K'') + \ell(K''; A_2)} [A_1 \sqcup K'] [K'' \sqcup A_2] = 0 \quad (\dagger\dagger)$$

with all the terms of the same degree. The proof concludes in exactly the same manner as the proof of Lemma 2.3.8(b). \square

Theorem 2.5.11. *Let $\delta = (c_1, c_2, \dots, c_n)$ and let V be the homogeneous component of $F_q(i_1, \dots, i_s; n)$ with degree δ and set $\gamma = (1^{m_1}, 2^{m_2}, \dots, n^{m_n})$.*

The products $[T]$ as (T) runs over all preferred tableaux with content γ form a basis for V .

Proof. We refer the reader to the proof of Theorem 2.3.9. \square

Corollary 2.5.12. *The products $[T]$ as (T) runs over all preferred tableaux form a basis for $F_q(i_1, \dots, i_s; n)$.* \square

This basis allows us to calculate the Gelfand-Kirillov dimension of the quantum flag variety. Consider the diagram of the partial ordering \leq_c on the minors of $F_q(i_1, \dots, i_s; n)$. A **saturated path** between two minors $a <_c b$ will be an ‘upwards path’ $a = a_1 <_c a_2 <_c \dots <_c a_l = b$ of minors such that no additional terms can be added; that is, for any index i there is no minor d such that $a_i <_c d <_c a_{i+1}$. The **length** of such a saturated path is defined to be l .

Example 2.5.13. An example of a saturated path between the minors $[134]$ and $[35]$ in $F_q(2, 3; 5)$ is

$$[134] <_c [234] <_c [235] <_c [245] <_c [24] <_c [25] <_c [35].$$

The length of this saturated path is 7.

A **maximal path** is a saturated path between the two minors $[1 \dots i_s]$ and $[n - i_1 + 1 \dots n]$. We claim that any maximal path will have length

$$\sum_{i=1}^s \{(i_t - i_{t-1})(n - i_t) + 1\}$$

where $i_0 = 0$. First we construct one example of a maximal path and show that it has the required length.

Consider the diagram of the partial ordering \leq_c on the minors of $F_q(i_1, \dots, i_s; n)$. Begin at the top of the diagram at the minor $[n - i_1 + 1 \dots n]$ and work down the diagram following the standard tower for $[n - i_1 + 1 \dots n]$ until we reach $[1 \dots i_1]$. It is clear that this will involve $i_1(n - i_1) + 1$ minors. Immediately beneath $[1 \dots i_1]$ we have the minor $[1 \dots i_1 \ n - i_2 + i_1 + 1 \dots n]$. Proceed down the diagram following the standard tower for $[1 \dots i_1 \ n - i_2 + i_1 + 1 \dots n]$ until we reach the minor $[1 \dots i_2]$. This will involve $(i_2 - i_1)(n - i_2) + 1$ minors. We repeat this process until we get to the minor $[1 \dots i_s]$. It is clear that when we reverse the sequence of minors described above, we obtain a maximal path in $F_q(i_1, \dots, i_s; n)$. We will call this maximal path the **standard maximal path** for $F_q(i_1, \dots, i_s; n)$. The length of the standard maximal path is clearly $\sum_{i=1}^s \{(i_t - i_{t-1})(n - i_t) + 1\}$.

An argument similar to that in the case of quantum Grassmannians establishes that the length of any maximal path is

$$\sum_{i=1}^s \{(i_t - i_{t-1})(n - i_t) + 1\},$$

as required.

Proposition 2.5.14. *Let α be the length of a maximal path in the diagram of the partial ordering \leq_* on the minors of $F_q(i_1, \dots, i_s; n)$. Then*

$$\text{GKdim}(F) = \alpha = \sum_{i=1}^s \{(i_t - i_{t-1})(n - i_t) + 1\}.$$

Proof. Once again, we refer the reader to the corresponding result for quantum Grassmannians, Proposition 2.3.14. \square

Example 2.5.15. Figure 2.4 shows the partial ordering \leq_c on $F := F_q(2, 3; 5)$ and the dotted lines show the route of the standard maximal path described above.

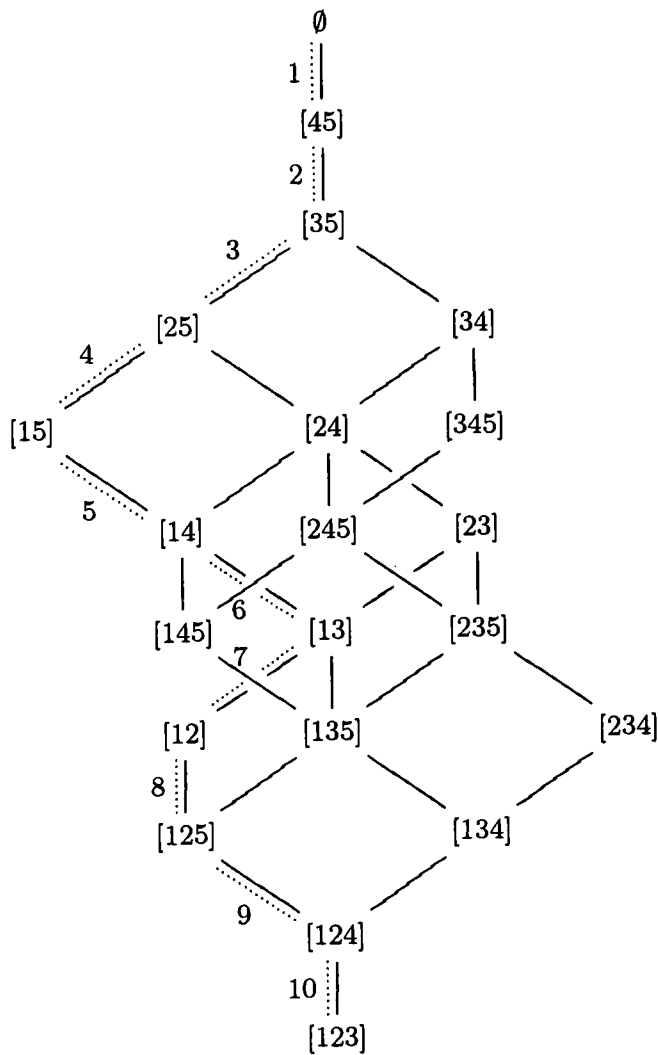


Figure 2.4: The partial ordering \leq_c on $F_q(2, 3; 5)$

Thus, $\text{GKdim}(F) = 10$ ($= (2(5 - 2) + (3 - 2)(5 - 3) + 1)$).

Chapter 3

Non-Commutative Dehomogenisation

Consider a commutative graded algebra R and let x be a regular homogeneous element of degree 1. The **dehomogenisation** of R at x is defined to be the factor ring

$$A := \frac{R}{\langle x - 1 \rangle},$$

where $\langle x - 1 \rangle$ is the ideal generated by $x - 1$ ([4] pg. 38). In this situation the algebra R and the dehomogenisation A are closely related, and we can hope to study R by first studying A . In particular, there is a bijective correspondence between the set of ideals $I \in \text{GrSpec} R$ such that x is normal modulo I , and the set of all ideals in the dehomogenisation.

In the non-commutative case, this definition of dehomogenisation becomes unsuitable. In this case, setting $x = 1$ can create too many relations and make $R / \langle x - 1 \rangle$ small and of little interest.

Example 3.0.1. Consider the quantum plane $k_q[x, y]$; that is, the k -algebra generated by indeterminates x, y subject to the relation $xy = qyx$, where $1 \neq q \in k$. The quantum plane has an obvious \mathbb{N} -grading where we take both x and y to have degree 1. It is clear that y is a regular normal element of degree 1, so factor $k_q[x, y]$ by the ideal generated by $y - 1$ and identify x with its image in the factor ring. Then

$$x = qx \Rightarrow x = 0$$

and the algebra ‘collapses’ to the field.

Returning to the commutative setting, there is an alternative description of dehomogenisation. As above, let R be a commutative graded algebra and x be a

regular homogeneous element of degree 1. Then we can also study $S := R_x$, the localisation of R at x . The algebra S has an induced \mathbb{Z} -grading from R and we have the following result.

Proposition 3.0.2. *Let R be a commutative graded algebra and x be a regular homogeneous element of degree 1. Let A denote the dehomogenisation of R at x and let S denote the localisation of R at x . Then*

$$A \cong S_0.$$

Proof. See [4] Proposition 1.5.18.

Let $\pi : R \rightarrow A$ be the natural map from R to A . We can factor π through a homomorphism $\psi : S \rightarrow A$. Then the kernel of ψ is $(x - 1)S$ and there is an induced isomorphism $S/(x - 1)S \cong A$. We also have that $S_0 \cong S/(x - 1)S$. Therefore $A \cong S_0$. \square

In this chapter we will use this alternative description of dehomogenisation to define the dehomogenisation of a non-commutative \mathbb{N} -graded algebra at a regular normal homogeneous element of degree 1. We then proceed to find a correspondence between those graded ideals of the ring, modulo which x is normal, and a certain subset of ideals of the dehomogenisation. This correspondence will be similar to the correspondence mentioned above in the commutative case. In Chapter 4 we will make extensive use of this correspondence in our hunt for the graded prime spectrum of the 2×4 quantum Grassmannian.

In Section 3.2 we turn our attention to a specific example of non-commutative dehomogenisation: the dehomogenisation of the $m \times n$ quantum Grassmannian at the ‘rightmost’ minor. In a reflection of a well known classical result, we show that this dehomogenisation is isomorphic to the coordinate ring of the $m \times (n - m)$ quantum matrices.

3.1 Non-commutative Dehomogenisation

Let R be a non-commutative \mathbb{N} -graded ring and let x be a regular normal homogeneous element of degree 1. Then by Example 1.4.22, the set $\{x^i \mid i \geq 0\}$ is a right Ore set in R and therefore R_x , the localisation of R at x , exists. Let $S := R_x$ and define

$$S_i := \sum_{l=0}^{\infty} R_{i+l} x^{-l}$$

for $i \in \mathbb{Z}$. Let σ be the automorphism obtained by conjugating by x . Notice that $xR_i = R_i x$ and so $\sigma(R_i) = R_i$. Then

$$\begin{aligned} S_i S_j &= \left(\sum_{l=0}^{\infty} R_{i+l} x^{-l} \right) \left(\sum_{m=0}^{\infty} R_{j+m} x^{-m} \right) \\ &= \sum_{l=0}^{\infty} \sum_{m=0}^{\infty} R_{i+l} x^{-l} R_{j+m} x^{-m} \\ &\subseteq \sum_{l=0}^{\infty} \sum_{m=0}^{\infty} R_{i+l} \sigma^{-l}(R_{j+m}) x^{-l-m} \\ &\subseteq \sum_{l=0}^{\infty} \sum_{m=0}^{\infty} R_{i+j+l+m} x^{-l-m} \\ &\subseteq \sum_{t=0}^{\infty} R_{i+j+t} x^{-t} \\ &= S_{i+j}. \end{aligned}$$

Thus the family of additive subgroups $\{S_i \mid i \in \mathbb{Z}\}$ of S forms a \mathbb{Z} -grading on S and

$$S = \dots \oplus S_{-1} \oplus S_0 \oplus S_1 \oplus \dots$$

In particular, note that $S_0 = \sum_{l=0}^{\infty} R_l x^{-l}$.

The commutative theory suggests that we make the following definition.

Definition 3.1.1. Let R be a non-commutative \mathbb{N} -graded ring and let x be a regular normal homogeneous element of degree 1. Let S be the localisation of R at x . We define the **dehomogenisation of R at x** :

$$\text{Dhom}(R, x) := S_0.$$

We can write R_x as a skew Laurent extension of $\text{Dhom}(R, x)$.

Lemma 3.1.2. *Let R be an \mathbb{N} -graded ring and let x be a regular normal homogeneous element of degree 1. Then*

$$R_x \cong \text{Dhom}(R, x)[y, y^{-1}; \sigma]$$

for some indeterminate y , where σ is the automorphism obtained by conjugating by x ; that is, $xr = \sigma(r)x$.

Proof. Consider the linear map

$$\begin{aligned} \theta : \text{Dhom}(R, x)[y, y^{-1}; \sigma] &\rightarrow S \\ r_m x^{-m} y^i &\mapsto r_m x^{-m+i} \end{aligned}$$

for $r_m \in R_m$, $m \in \mathbb{N}$ and $i \in \mathbb{Z}$. Then θ is an additive map which is injective and onto. Let $r_m \in R_m$, $r_n \in R_n$, $m, n \in \mathbb{N}$ and $i, j \in \mathbb{Z}$. Then

$$\begin{aligned} \theta((r_m x^{-m} y^i)(r_n x^{-n} y^j)) &= \theta(r_m x^{-m} \sigma^i(r_n) \sigma^i(x^{-n}) y^{i+j}) \\ &= \theta(r_m x^{-m} \sigma^i(r_n) x^{-n} y^{i+j}) \\ &= \theta(r_m \sigma^{i-m}(r_n) x^{-m-n} y^{i+j}) \\ &= r_m \sigma^{i-m}(r_n) x^{i+j-(m+n)} \\ &= r_m x^{i-m} r_n x^{j-n} \\ &= \theta(r_m x^{-m} y^i) \theta(r_n x^{-n} y^j), \end{aligned}$$

as required. □

Lemma 3.1.2 proves to be useful in proving some relationships between properties of the ring and properties of the dehomogenisation of the ring at a regular normal homogeneous element.

Corollary 3.1.3. *Let R be an \mathbb{N} -graded noetherian ring and let x be a regular normal homogeneous element of degree 1. If $\text{Dhom}(R, x)$ is a domain, then R is a domain.* □

Proposition 3.1.4. *Let R be an \mathbb{N} -graded noetherian ring and let x be a regular normal homogeneous element of degree 1. Then R_x and $\text{Dhom}(R, x)$ are also noetherian.*

Proof. Let $S = R_x$ and let I be a non-zero right ideal of S . We will show that I is finitely generated, thereby proving that S is noetherian. Let $0 \neq \alpha \in I$. Then $\alpha = rx^{-t}$ for some $t \in \mathbb{Z}$. Therefore

$$\alpha x^t = r \in I \cap R$$

and

$$\alpha = (\alpha x^t) x^{-t} \in (I \cap R) S.$$

Therefore $I = (I \cap R) S$. Now R is noetherian and therefore finitely generated, thus I is finitely generated and S is noetherian.

Now suppose that $\text{Dhom}(R, x)$ is not noetherian. Then by Lemma 3.1.2, S is not noetherian, which is a contradiction. Therefore $\text{Dhom}(R, x)$ is noetherian. \square

The first part of the following result is well known and can be found in [17] Lemma 3.3.

Proposition 3.1.5. *Let R be a finitely generated \mathbb{N} -graded algebra with a regular normal homogeneous element x of degree 1. Let σ be the automorphism obtained by conjugating by x . Suppose that R has a finite dimensional generating subspace V containing 1 such that $\sigma(V) = V$. Then*

- (i) $\text{GKdim}(R_x) = \text{GKdim}(R)$;
- (ii) $\text{GKdim}(\text{Dhom}(R, x)) = \text{GKdim}(R) - 1$.

Proof.

(i) See [17] Lemma 3.3.

(ii) By Lemma 3.1.2, $R_x \cong \text{Dhom}(R, x)[y, y^{-1}; \sigma]$, where σ is the automorphism obtained by conjugating by x . Now

$$\text{Dhom}(R, x)[y, y^{-1}; \sigma] \cong \text{Dhom}(R, x)[y, \sigma]_y.$$

Therefore, by Proposition 1.3.5 and (i) above,

$$\begin{aligned} \text{GKdim}(\text{Dhom}(R, x)[y, y^{-1}; \sigma]) &= \text{GKdim}(\text{Dhom}(R, x)[y, \sigma]_y) \\ &= \text{GKdim}(\text{Dhom}(R, x)) + 1. \end{aligned}$$

Therefore $\text{GKdim}(\text{Dhom}(R, x)) = \text{GKdim}(R_x) - 1 = \text{GKdim}(R) - 1$. \square

The remainder of this section is devoted to finding the correspondence between the graded ideals of a noetherian ring R and the ideals in the dehomogenisation of the ring at a regular normal homogeneous element of degree 1. Let us return for a moment to the commutative setting where we have the following result ([4], Exercise 1.5.26).

Proposition 3.1.6. *Let R be a commutative graded ring and let x be a regular homogeneous element of degree 1. Let A be the dehomogenisation of R at x and*

S be the localisation of R at x and identify A with S_0 . If $\pi : R \rightarrow A$ is the natural homomorphism, we have

(a) $\pi(I) = IS \cap A$ for every graded ideal I of R and $J = \pi(JS \cap R)$ for every ideal J of A ;

(b) π induces a bijective correspondence between those graded ideals of R , modulo which x is regular, and the set of all ideals of A ;

(c) the above correspondence preserves inclusions and intersections and the property of being prime. \square

The correspondence in our case should follow the above closely. However, we will need to restrict to a certain subset of the ideals in the dehomogenisation.

Definition 3.1.7. Let R be a ring and $\sigma : R \rightarrow R$ be an automorphism of R . An ideal I of R is called a σ -ideal if $\sigma(I) \subseteq I$. In particular, if R is noetherian and I is a σ -ideal of R , then $\sigma(I) = I$. An ideal P of R is σ -prime if P is a proper σ -ideal of R and whenever A, B are σ -ideals of R such that $AB \subseteq P$, then $A \subseteq P$ or $B \subseteq P$. We denote the set of σ -prime ideals of R by $\sigma\text{-Spec}R$.

The correspondence that we find will be a homeomorphism with respect to the Zariski topology on $\text{Spec}R$. Recall the following definition (see for example, [23] Chapter 12).

Definition 3.1.8. For any ideal I of R , let

$$V(I) = \{P \in \text{Spec}R \mid I \subseteq P\},$$

$$W(I) = \{P \in \text{Spec}R \mid I \not\subseteq P\}.$$

Then $\mathcal{W} = \{W(I) \mid I \text{ is an ideal of } R\}$ is closed under finite intersections and arbitrary unions, $\text{Spec}R = W(0)$ and $W(0) = \emptyset$. Thus \mathcal{W} is the family of open sets for a topology on $\text{Spec}R$ and the closed sets are precisely $V(I)$. This topology is the **Zariski topology**.

There is an induced topology on $\text{GrSpec}R$, the graded prime spectrum of R . For each graded ideal I define

$$V_{\text{gr}}(I) = \{P \in \text{GrSpec}R \mid I \subseteq P\},$$

$$W_{\text{gr}}(I) = \{P \in \text{GrSpec}R \mid I \not\subseteq P\}.$$

Then we have a topology on $\text{GrSpec}R$ with closed sets $V_{\text{gr}}(I)$ and open sets $W_{\text{gr}}(I)$. Similarly, we have an induced topology on the σ -prime spectrum of R with closed sets

$$V_{\sigma}(I) = \{P \in \sigma\text{-Spec}R \mid I \subseteq P\},$$

and open sets

$$W_{\sigma}(I) = \{P \in \sigma\text{-Spec}R \mid I \not\subseteq P\}$$

for a σ -ideal I of R .

Let R be an \mathbb{N} -graded noetherian ring and x be a regular normal homogeneous element in R of degree 1. Let $S := R_x$. Then $\text{Dhom}(R, x) = S_0$. Let I be a graded ideal of R . Then we have the following chain of ideals:

$$I \mapsto IS \mapsto IS \cap S_0. \quad (\dagger)$$

Let $\sigma_x : S \rightarrow S$ be the automorphism obtained by conjugating by x ; that is, $\sigma_x(s) = x^{-1}sx$ for $s \in S$. Note that σ_x restricted to S_0 is an automorphism. We will show that the chain of ideals (\dagger) defines a map Γ_x from the set of graded ideals of R modulo which x is normal, to the set of proper σ_x -ideals of S_0 . Furthermore, we will show that the restriction of Γ_x to the relevant graded prime ideals of R not containing x is a homeomorphism from $W_{\text{gr}}(\langle x \rangle)$ to $\sigma_x\text{-Spec}S_0$ with respect to the Zariski topology.

Lemma 3.1.9. *Let Y be the set of graded ideals of R modulo which x is regular and let $I \in Y$. The chain of ideals*

$$I \longrightarrow IS \longrightarrow IS \cap S_0$$

defines a map

$$\Gamma_x : Y \longrightarrow \{\text{proper } \sigma_x\text{-ideals of } S_0\}$$

such that Γ_x is:

(i) inclusion preserving;

(ii) one to one;

(iii) onto.

Proof. Let $I \in Y$. Then by Proposition 1.4.32, IS is an ideal of S . In fact since R/I is x -torsion free we have that $I = IS \cap R$ and I is a proper ideal

of R . Therefore IS is a proper ideal of S . Also, I is a graded ideal, therefore $I = \oplus (I \cap R_d)$ and

$$IS = (\oplus (I \cap R_d)) S \Rightarrow IS = \oplus (IS \cap S_i).$$

Thus IS is a proper graded ideal of S .

Now, $\sigma_x(IS) \subseteq xISx^{-1} \subseteq IS$, and since σ_x restricts to an automorphism on S_0 we have that $IS \cap S_0$ is a σ_x -ideal of S_0 . Suppose that $IS \cap S_0 = S_0$. Then $S_0 \subseteq IS$ and $1 \in IS$, contradicting IS being a proper ideal of S . Therefore $\Gamma_x(I) = IS \cap S_0$ is a proper σ_x -ideal of S_0 .

(i) Let $I, J \in Y$ be such that $I \subseteq J$. Certainly $IS \subseteq JS$ and $IS \cap S_0 \subseteq JS \cap S_0$; that is, $\Gamma_x(I) \subseteq \Gamma_x(J)$ and Γ_x is inclusion preserving.

(ii) Let $I, J \in Y$ such that $I \neq J$. Then $IS \neq JS$. Let $y \in I$ be homogeneous of degree t such that $y \notin J$. Then $yx^{-t} \in IS \cap S_0$, but $yx^{-t} \notin JS$. Therefore $\Gamma_x(I) \neq \Gamma_x(J)$ and Γ_x is one to one.

(iii) Let T be a proper σ_x -ideal of S_0 . Then TS is a graded ideal of S , since if $s_i x^{-j} \in S_i$, then $s_i x^{-j}(TS) = s_i x^{-j-i}(x^i TS) \subseteq S_0 \sigma_x^i(T)S \subseteq S_0 TS \subseteq TS$. By Proposition 1.4.31, $TS \cap R$ is an ideal of R . In fact $TS \cap R$ is a graded ideal of R and x is regular modulo $TS \cap R$, since if $rx \in TS \cap R$ for some $r \in R$, then $r \in TS$. Finally,

$$\Gamma_x(TS \cap R) = (TS \cap R)S \cap S_0 = TS \cap S_0 = T.$$

□

Let $P \in W_{\text{gr}}(\langle x \rangle)$. Then since x is normal, x is regular modulo P and we have the following Lemma.

Lemma 3.1.10. *Let $P \in W_{\text{gr}}(\langle x \rangle)$. Then $\Gamma_x(P) \in \sigma_x\text{-Spec}S_0$.*

Proof. Let $P \in W_{\text{gr}}(\langle x \rangle) = \{Q \in \text{GrSpec}(R) \mid \langle x \rangle \not\subseteq Q\}$. By Lemma 3.1.9, $\Gamma_x(P)$ is a proper σ_x -ideal of S_0 . Consider PS . By the proof of Lemma 3.1.9, PS is a proper graded ideal of S . Now, since R/P is x -torsion free, we have from Proposition 1.4.29 that $P = PS \cap R$. Thus by Proposition 1.4.31, PS is a prime ideal of S . It follows from Corollary 1.2.6 that $P \in \text{GrSpec}(S)$. Let A, B be σ_x -ideals of S_0 such that $AB \subseteq PS \cap S_0$. Then

$$AB \subseteq PS \Rightarrow (AS)(BS) \subseteq PS.$$

Therefore, without loss of generality, $AS \subseteq PS$. Thus $AS \cap S_0 \subseteq PS \cap S_0$ and $A \subseteq PS \cap S_0$; that is, $\Gamma_x(P) \in \sigma_x\text{-Spec}S_0$. □

In the third part of the proof of Lemma 3.1.9 an inverse for Γ_x has been constructed:

$$\Gamma_x^{-1} : \{\text{proper } \sigma_x\text{-ideals of } S_0\} \longrightarrow Y$$

given by $\Gamma_x^{-1}(T) = TS \cap R$ for $T \in \{\text{proper } \sigma_x\text{-ideals of } S_0\}$. The following Proposition proves that when we restrict Γ_x to $W_{\text{gr}}(\langle x \rangle)$ (and Γ_x^{-1} to $\sigma_x\text{-Spec}(S_0)$) we obtain a homeomorphism with respect to the Zariski topology.

Proposition 3.1.11.

$$\begin{aligned} \Gamma_x : W_{\text{gr}}(\langle x \rangle) &\rightarrow \sigma_x\text{-Spec}S_0 \\ P &\mapsto PS \cap S_0 \end{aligned}$$

is a homeomorphism with respect to the Zariski topology.

Proof. Lemma 3.1.9 gives us that Γ_x is one to one and inclusion preserving. Let $T \in \sigma_x\text{-Spec}S_0$ and consider $TS \cap R$, which by the proof of Lemma 3.1.9 is a graded ideal of R , modulo which x is normal. We show that $TS \cap R \in W_{\text{gr}}(\langle x \rangle)$. Suppose that A, B are graded ideals of S with $AB \subseteq TS$. Then

$$(A \cap S_0)(B \cap S_0) \subseteq (AB) \cap S_0 \subseteq (TS) \cap S_0$$

Since $A \cap S_0$ and $B \cap S_0$ are σ_x -ideals of S_0 , without loss of generality, $A \cap S_0 \subseteq T$. Thus $A = (A \cap S_0)S \subseteq TS$ and $TS \in \text{GrSpec}S$. Therefore, by Theorem 1.4.33, $TS \cap R$ is a graded prime ideal of R disjoint from $\{x^i \mid i \in \mathbb{N}\}$, and therefore $TS \cap R \in W_{\text{gr}}(\langle x \rangle)$.

We have shown that Γ_x is inclusion preserving, one to one and onto, and constructed the map

$$\begin{aligned} \Gamma_x^{-1} : \sigma_x\text{-Spec}S_0 &\rightarrow W_{\text{gr}}(\langle x \rangle) \\ T &\mapsto TS \mapsto TS \cap R. \end{aligned}$$

It is clear that Γ_x^{-1} is also inclusion preserving. It remains to show that both Γ_x and Γ_x^{-1} are continuous.

Let $W_{\text{ind}}(I) = \{P \in W_{\text{gr}}(\langle x \rangle) \mid I \not\subseteq P\}$, for any graded ideal I of R . Let J be a proper σ_x -ideal of S_0 and consider the open set $W_{\sigma_x}(J)$ in $\sigma_x\text{-Spec}S_0$. Then

$$Q \in W_{\sigma_x}(J) \Rightarrow J \not\subseteq Q \Rightarrow \Gamma_x^{-1}(J) \not\subseteq \Gamma_x^{-1}(Q).$$

Therefore $\Gamma_x^{-1}(Q) \in W_{\text{ind}}(\Gamma_x^{-1}(J))$ and thus $\Gamma_x^{-1}(W_{\sigma_x}(J)) \subseteq W_{\text{ind}}(\Gamma_x^{-1}(J))$.

Now let $T \in W_{\text{ind}}(\Gamma_x^{-1}(J))$. Then

$$T \notin \Gamma_x^{-1}(J) \Rightarrow \Gamma_x(T) \notin \Gamma_x(\Gamma_x^{-1}(J)) = J.$$

Therefore $\Gamma_x(T) \in W_{\sigma_x}(J)$ and $\Gamma_x(W_{\text{ind}}(\Gamma_x^{-1}(J))) \subseteq W_{\sigma_x}(J)$; thus we have the reverse inequality, $W_{\text{ind}}(\Gamma_x^{-1}(J)) \subseteq \Gamma_x^{-1}(W_{\sigma_x}(J))$. Therefore

$$\Gamma_x^{-1}(W_{\sigma_x}(J)) = W_{\text{ind}}(\Gamma_x^{-1}(J))$$

and Γ_x is continuous. \square

In Chapter 4 we will need to use the map Γ_x^{-1} to explicitly find graded primes in $G_q(2, 4)$.

Proposition 3.1.12. *Let $R = \bigoplus_{i \geq 0} R_i$ be a connected \mathbb{N} -graded noetherian k -algebra (so $R_0 = k$) and let x be a regular normal homogeneous element in R of degree 1. Let $S := R_x$ and suppose that $R = k[R_1]$; that is, R is generated over k by R_1 . Then*

$$S_0 = k[R_1 x^{-1}];$$

that is, S_0 is generated over k by $R_1 x^{-1}$.

Proof. Let $s \in S_0$. Then $s = s_0 + s_1 x^{-1} + \dots + s_n x^{-n}$ for some $s_i \in R_i$. Consider $s_i x^{-i}$. Since $R = k[R_1]$,

$$s_i = \sum \lambda_j r_{j_1} \dots r_{j_i} \text{ for some } \lambda_j \in k, r_{j_i} \in R_1.$$

So

$$\begin{aligned} s_i x^{-i} &= \sum \lambda_j r_{j_1} \dots r_{j_i} x^{-i} \\ &= \sum \lambda_j r_{j_1} x^{-1} \sigma^{-1}(r_{j_2}) x^{-1} \dots \sigma^{-i+1}(r_{j_i}) x^{-1} \\ &\in k[R_1 x^{-1}]. \end{aligned}$$

Therefore $s \in k[R_1 x^{-1}]$ and since the reverse inequality is trivial, we have that

$$S_0 = k[R_1 x^{-1}].$$

\square

3.2 Dehomogenisation of $G_q(n, 2n)$ at $[n + 1 \dots 2n]$

In the classical theory, a well known result is that the dehomogenisation of the coordinate ring of the $n \times 2n$ Grassmannian at the minor $[n + 1 \dots 2n]$ is isomorphic to the coordinate ring of the $n \times n$ matrices; that is,

$$\frac{\mathcal{O}(G(n, 2n))}{\langle [n + 1 \dots 2n] - 1 \rangle} \cong \mathcal{O}(M_n(k)).$$

Having given a suitable definition of non-commutative dehomogenisation, it is natural to ask if we have a similar result in the quantum case. We will consider $G_q(n, 2n)$ as an \mathbb{N} -graded k -algebra with each $n \times n$ minor defined to have degree 1. Our aim is to dehomogenise $G_q(n, 2n)$ at $[n + 1 \dots 2n]$ and show that this dehomogenisation is isomorphic to $\mathcal{O}_q(M_n)$. First we must check that $[n + 1 \dots 2n]$ is normal in $G_q(n, 2n)$.

Let $I \subseteq \{1, \dots, 2n\}$ with $|I| = n$. Then from [7] Lemma 2.11 and Theorem 3.6, we obtain the following commutation relation in $\mathcal{G}_h(n, 2n)$:

$$[I][n + 1 \dots 2n] = h^{n-a} [n + 1 \dots 2n][I]$$

where $a = |I \cap \{n + 1, \dots, 2n\}|$. We use this relation to prove that $[n + 1 \dots 2n]$ is normal in $G_q(n, 2n)$.

Lemma 3.2.1. *Let $I \subseteq \{1, \dots, 2n\}$ with $|I| = n$. Then*

$$[I][n + 1 \dots 2n] = q^{n-a} [n + 1 \dots 2n][I]$$

where $a = |I \cap \{n + 1, \dots, 2n\}|$ and thus $[n + 1 \dots 2n]$ is normal in $G_q(n, 2n)$.

Proof. Recall the homomorphism

$$\begin{aligned} \theta : \mathcal{G}_h(n, 2n) &\rightarrow G_q(n, 2n) \\ h &\mapsto q \end{aligned}$$

from the proof of Lemma 2.2.5. In $\mathcal{G}_h(n, 2n)$ we have the commutation relation

$$[I][n + 1 \dots 2n] = h^{n-a} [n + 1 \dots 2n][I],$$

and since $[n + 1 \dots 2n][I] \notin \ker \theta$ it follows that

$$[I][n + 1 \dots 2n] = q^{n-a} [n + 1 \dots 2n][I].$$

□

Therefore the $n \times 2n$ quantum Grassmannian is an \mathbb{N} -graded k -algebra and $[n+1 \dots 2n]$ is a regular normal element of degree 1 in $G_q(n, 2n)$. Thus we may localise at $[n+1 \dots 2n]$:

$$S := G_q(n, 2n)_{[n+1 \dots 2n]}.$$

There is an induced \mathbb{Z} -grading on S and $\text{Dhom}(G_q(n, 2n), [n+1 \dots 2n]) = S_0$ is generated by elements of the form

$$\{I\} := [I] [n+1 \dots 2n]^{-1}.$$

Before proceeding we establish some notation.

Given a set $I = \{i_1, \dots, i_n\} \subseteq \{1, \dots, 2n\}$ we will denote the set with i_k omitted by $\{i_1, \dots, \widehat{i_k}, \dots, i_n\}$. Given two sets $I, J \subseteq \{1, \dots, 2n\}$ we define

$$\ell(I : J) := |\{(i, j) \in I \times J : i > j\}|.$$

The following Lemma allows us to expand quantum determinants in $\mathcal{O}_q(M_n)$ along rows or down columns. These relations are special cases of the Laplace expansions for quantum matrices (see [36] Section 4.4).

Lemma 3.2.2. *Let $i, k \leq n$. Then*

$$\begin{aligned} \delta_{ik} D &= \sum_{j=1}^n (-q)^{j-k} X_{ij} A(kj) = \sum_{j=1}^n (-q)^{i-j} A(ij) X_{kj} \\ &= \sum_{j=1}^n (-q)^{j-k} X_{ij} A(jk) = \sum_{j=1}^n (-q)^{i-j} A(ji) X_{jk} \end{aligned}$$

where $A(kj)$ is the quantum minor in $\mathcal{O}_q(M_n)$ obtained by deleting row k and column j and δ_{ik} is the Kronecker delta. \square

Recall that if $\phi : R \rightarrow R$ is an anti-endomorphism on a ring R , then $\phi(rs) = \phi(s)\phi(r)$ for $r, s \in R$. We will require the following anti-endomorphism on $\mathcal{O}_q(M_n)$ given in [36] Corollary 5.2.2:

$$\begin{aligned} \Gamma : \mathcal{O}_q(M_n) &\rightarrow \mathcal{O}_q(M_n) \\ X_{ij} &\mapsto (-q)^{i-j} A(ji) \end{aligned}$$

where $A(ji)$ is the $(n-1) \times (n-1)$ minor in $\mathcal{O}_q(M_n)$ obtained by removing row j and column i . It will be necessary for us to know what Γ will do to a minor in $\mathcal{O}_q(M_n)$.

Lemma 3.2.3. Let $1 \leq i_1 < i_2 < \dots i_r \leq n$ and $1 \leq j_1 < j_2 < \dots j_r \leq n$. Then

$$\Gamma([i_r \dots i_1 | j_1 \dots j_r]) = (-q)^{\sum_{s=1}^r i_s - j_s} D^{r-1} \left[n \dots \widehat{j_r} \dots \widehat{j_1} \dots 1 | 1 \dots \widehat{i_1} \dots \widehat{i_r} \dots n \right].$$

Proof. We use induction on r .

In the case that $r = 1$ the claim is true by definition. Now let $r > 1$ and suppose that the result is true for all $k < r$. Then

$$\begin{aligned} & \Gamma([i_r \dots i_1 | j_1 \dots j_r]) \\ &= \Gamma \left(\sum_{k=1}^r (-q)^{1-k} \left[i_r \dots \widehat{i_k} \dots i_1 | \widehat{j_1} j_2 \dots j_r \right] X_{i_k, j_1} \right) \\ &= \sum_{k=1}^r (-q)^{1-k} \Gamma(X_{i_k, j_1}) \Gamma \left(\left[i_r \dots \widehat{i_k} \dots i_1 | \widehat{j_1} j_2 \dots j_r \right] \right) \\ &= \sum_{k=1}^r (-q)^{1-k} (-q)^{i_k - j_1} (-q)^{\sum_{s \neq k} i_s - \sum_{s \neq 1} j_s} A(j_1 i_k) \\ & \quad \times D^{r-2} \left[n \dots \widehat{j_r} \dots \widehat{j_2} \dots 1 | 1 \dots \widehat{i_1} \dots i_k \dots \widehat{i_r} \dots n \right] \\ &= (-q)^{\sum i_s - j_s} D^{r-2} \sum_{k=1}^r (-q)^{1-k} A(j_1 i_k) \\ & \quad \times \left[n \dots \widehat{j_r} \dots \widehat{j_2} \dots 1 | 1 \dots \widehat{i_1} \dots i_k \dots \widehat{i_r} \dots n \right] \\ &= (-q)^{\sum i_s - j_s} D^{r-2} \sum_{k=1}^r (-q)^{1-k} A(j_1 i_k) \\ & \quad \times \sum_{l \in S} (-q)^{-i_k + (k-1) + \tilde{l}} X_{l, i_k} \left[n \dots \widehat{l j_r} \dots \widehat{j_2} \dots 1 | 1 \dots \widehat{i_1} \dots \widehat{i_r} \dots n \right] \\ &= (-q)^{\sum i_s - j_s} D^{r-2} \sum_{l \in S} \left(\sum_{k=1}^r (-q)^{-i_k + j_1} A(j_1 i_k) X_{l, i_k} \right) \\ & \quad \times (-q)^{-j_1 + \tilde{l}} \left[n \dots \widehat{l j_r} \dots \widehat{j_2} \dots 1 | 1 \dots \widehat{i_1} \dots \widehat{i_r} \dots n \right] \end{aligned}$$

where $S = \{1, \dots, \widehat{j_2}, \dots, \widehat{j_r}, \dots, n\}$ and $\tilde{l} = \ell(\{l\}, \{1, \dots, \widehat{j_2}, \dots, \widehat{j_r}, \dots, n\})$.

By Lemma 3.2.2

$$\begin{aligned}
& \sum_{k=1}^r (-q)^{-i_k+j_1} A(j_1 i_k) X_{l, i_k} \\
&= \sum_{m=1}^n (-q)^{-m+j_1} A(j_1 m) X_{lm} - \sum_{m \neq i_s} (-q)^{-m+j_1} A(j_1 m) X_{lm} \\
&= \delta_{j_1 l} D - \sum_{m \neq i_s} (-q)^{-m+j_1} A(j_1 m) X_{lm}
\end{aligned}$$

where $\delta_{j_1 l}$ is the Kronecker delta. Therefore,

$$\begin{aligned}
& \Gamma([i_r \dots i_1 | j_1 \dots j_r]) \\
&= (-q)^{\sum i_s - j_s} D^{r-2} \sum_{l \in S} \left(\delta_{j_1 l} D - \sum_{m \neq i_s} (-q)^{-m+j_1} A(j_1 m) X_{lm} \right) \\
&\quad \times (-q)^{-j_1 + \bar{l}} [n \dots \widehat{l} \widehat{j_r} \dots \widehat{j_2} \dots 1 | 1 \dots \widehat{i_1} \dots \widehat{i_r} \dots n] \\
&= (-q)^{\sum i_s - j_s} D^{r-2} \sum_{l \in S} \delta_{j_1 l} D (-q)^{-j_1 + \bar{l}} \\
&\quad \times [n \dots \widehat{l} \widehat{j_r} \dots \widehat{j_2} \dots 1 | 1 \dots \widehat{i_1} \dots \widehat{i_r} \dots n] \\
&= (-q)^{\sum i_s - j_s} D^{r-1} [n \dots \widehat{j_r} \dots \widehat{j_1} \dots 1 | 1 \dots \widehat{i_1} \dots \widehat{i_r} \dots n].
\end{aligned}$$

The penultimate equality here is due to Lemma 3.2.2:

$$\sum_{l \in S} (-q)^{\bar{l} - (i_1 - 1)} X_{lm} [1 \dots j_1 \widehat{j_2} \dots \widehat{j_r} \widehat{l} \dots n] = \delta_{mi_1} D = 0.$$

□

Let A be the k -subalgebra of $\text{Dhom}(G_q(n, 2n), [n+1 \dots 2n]) = S_0$ generated by the set $\{\{j \mid n+1 \dots \widehat{i} \dots 2n\} : 1 \leq j \leq n < i \leq 2n\}$. In Proposition 3.2.4 we show that there exists a map from the $n \times n$ quantum matrices onto A . In Lemma 3.2.5 we prove that the subalgebra A is actually the whole of S_0 and therefore we have a map from $\mathcal{O}_q(M_n)$ onto S_0 . We should note that in [9] an isomorphism between the quantum matrices $\mathcal{O}_q(M_n)$, and A is established. However in [9], there is no proof that A is actually the whole of the dehomogenisation, S_0 .

Proposition 3.2.4. *There is a homomorphism ρ from $\mathcal{O}_q(M_n)$ onto A defined by*

$$\begin{aligned}\rho : \mathcal{O}_q(M_n) &\rightarrow A \\ X_{2n+1-i,j} &\mapsto \{j \ n+1 \dots \widehat{i} \dots 2n\},\end{aligned}$$

for $1 \leq j \leq n < i \leq 2n$.

Proof. It is clear that the map ρ is onto. In order to show that it is a homomorphism we have to show that the images of the X_{ij} under ρ still obey the relevant commutation relations. There are four types of products to be considered.

(1) Let $1 \leq i < l \leq n$ and $1 \leq j \leq n$. Then

$$X_{ij}X_{lj} = qX_{lj}X_{ij},$$

and we must show that

$$\rho(X_{ij})\rho(X_{lj}) = q\rho(X_{lj})\rho(X_{ij}).$$

Let $t = 2n + 1 - i$ and $s = 2n + 1 - l$ and consider the product

$$[j \ n+1 \dots \widehat{t} \dots 2n] [j \ n+1 \dots \widehat{s} \dots 2n]$$

in $G_q(n, 2n)$. We can think of this as a product in $\mathcal{O}_q(M_{n+1})$ and by applying the anti-endomorphism Γ to the commutation relation $X_{s,n+1}X_{t,n+1} = qX_{t,n+1}X_{s,n+1}$ we obtain:

$$\begin{aligned}[j \ n+1 \dots \widehat{t} \dots 2n] [j \ n+1 \dots \widehat{s} \dots 2n] \\ = q [j \ n+1 \dots \widehat{s} \dots 2n] [j \ n+1 \dots \widehat{t} \dots 2n].\end{aligned}$$

Multiplying through by $[n+1 \dots 2n]^{-2}$ on each side gives

$$\begin{aligned}\{j \ n+1 \dots \widehat{t} \dots 2n\} \{j \ n+1 \dots \widehat{s} \dots 2n\} \\ = q \{j \ n+1 \dots \widehat{s} \dots 2n\} \{j \ n+1 \dots \widehat{t} \dots 2n\};\end{aligned}$$

that is, $\rho(X_{ij})\rho(X_{lj}) = q\rho(X_{lj})\rho(X_{ij})$.

(2) Let $1 \leq j < r \leq n$ and $1 \leq i \leq n$. Then

$$X_{ij}X_{ir} = qX_{ir}X_{ij}.$$

Let $t = 2n + 1 - i$ and, as in (1), think of the product

$$[j \ n + 1 \dots \widehat{t} \dots 2n] [r \ n + 1 \dots \widehat{t} \dots 2n]$$

as sitting inside $\mathcal{O}_q(M_{n+1})$. Then Γ applied to the relation $X_{j,n+1}X_{r,n+1} = qX_{r,n+1}X_{j,n+1}$ gives us

$$\begin{aligned} & [j \ n + 1 \dots \widehat{t} \dots 2n] [r \ n + 1 \dots \widehat{t} \dots 2n] \\ & = q [r \ n + 1 \dots \widehat{t} \dots 2n] [j \ n + 1 \dots \widehat{t} \dots 2n]. \end{aligned}$$

Therefore, multiplying through by $[n + 1 \dots 2n]^{-2}$ we get

$$\begin{aligned} & \{j \ n + 1 \dots \widehat{t} \dots 2n\} \{r \ n + 1 \dots \widehat{t} \dots 2n\} \\ & = q \{r \ n + 1 \dots \widehat{t} \dots 2n\} \{j \ n + 1 \dots \widehat{t} \dots 2n\}. \end{aligned}$$

(3) Let $1 \leq i < l < j < r \leq n$. Then

$$X_{ij}X_{lr} = X_{lr}X_{ij} + (q - q^{-1}) X_{ir}X_{lj}.$$

Let $t = 2n + 1 - i$ and $s = 2n + 1 - l$ and consider the product

$$[j \ n + 1 \dots \widehat{t} \dots 2n] [r \ n + 1 \dots \widehat{s} \dots 2n]$$

as a product in $\mathcal{O}_q(M_{n+2})$. By Example 2.2.3 (and the symmetry of relations in $\mathcal{O}_q(M_{n+2})$)

$$[js \mid I] [rt \mid I] = [rt \mid I] [js \mid I] + (q - q^{-1}) [jt \mid I] [rs \mid I] \quad (\dagger)$$

where $I = \{n + 1, n + 2\}$. Let \mathcal{D} be the quantum determinant for the copy of $\mathcal{O}_q(M_{n+2})$ we are interested in; that is,

$$\mathcal{D} = [n + 2 \ n + 1 \ \dots \ 1 | j \ r \ n + 1 \ n + 2 \ \dots \ 2n].$$

Then by applying Γ to (\dagger) we obtain

$$\begin{aligned} & \mathcal{D} [r \ n + 1 \dots \widehat{s} \dots 2n] \mathcal{D} [j \ n + 1 \dots \widehat{t} \dots 2n] \\ & = \mathcal{D} [j \ n + 1 \dots \widehat{t} \dots 2n] \mathcal{D} [r \ n + 1 \dots \widehat{s} \dots 2n] \\ & \quad - (q - q^{-1}) \mathcal{D} [r \ n + 1 \dots \widehat{t} \dots 2n] \mathcal{D} [j \ n + 1 \dots \widehat{s} \dots 2n] \end{aligned}$$

where each of the minors in this relation involves rows $1, \dots, n$ of the $(n+2) \times (n+2)$ quantum matrix. Since \mathcal{D} is central in the domain $\mathcal{O}_q(M_{n+2})$, we can cancel \mathcal{D}^2 from each side and multiplying through by $[n+1 \dots 2n]^{-2}$ we get

$$\begin{aligned} & \{r \ n+1 \dots \widehat{s} \dots 2n\} \{j \ n+1 \dots \widehat{t} \dots 2n\} \\ &= \{j \ n+1 \dots \widehat{t} \dots 2n\} \{r \ n+1 \dots \widehat{s} \dots 2n\} \\ &\quad - (q - q^{-1}) \{r \ n+1 \dots \widehat{t} \dots 2n\} \{j \ n+1 \dots \widehat{s} \dots 2n\}. \end{aligned}$$

(4) Let $1 \leq i < l < j < r \leq n$. Then

$$X_{ir}X_{lj} = X_{lj}X_{ir}.$$

Let $t = 2n+1-i$ and $s = 2n+1-l$ and as in (3), applying Γ to the relation

$$[js \mid n+1 \ n+2][rt \mid n+1 \ n+2] = [rt \mid n+1 \ n+2][js \mid n+1 \ n+2]$$

in $\mathcal{O}_q(M_{n+2})$ gives us

$$\begin{aligned} & [r \ n+1 \dots \widehat{s} \dots 2n] [j \ n+1 \dots \widehat{t} \dots 2n] \\ &= [j \ n+1 \dots \widehat{t} \dots 2n] [r \ n+1 \dots \widehat{s} \dots 2n]. \end{aligned}$$

Multiplying through by u^{-2} we get

$$\begin{aligned} & \{r \ n+1 \dots \widehat{s} \dots 2n\} \{j \ n+1 \dots \widehat{t} \dots 2n\} \\ &= \{j \ n+1 \dots \widehat{t} \dots 2n\} \{r \ n+1 \dots \widehat{s} \dots 2n\}. \end{aligned}$$

□

Therefore we have a homomorphism ρ from the algebra of $n \times n$ quantum matrices onto a subalgebra of S_0 . The following Lemma shows that ρ actually maps onto the whole of S_0 .

Lemma 3.2.5. *The k -algebra S_0 is generated by the set $\{\{j \ n+1 \dots \widehat{i} \dots 2n\} : 1 \leq j \leq n < i \leq 2n\}$; that is,*

$$S_0 = A.$$

Proof. We show that any minor $\{i_1 \dots i_n\} \in S_0$ can be expressed as a k -linear combination of products of elements of the form

$$\{j \ n+1 \dots \widehat{i} \dots 2n\}$$

where $1 \leq j \leq n < i \leq 2n$.

Let $\{i_1 \leq \dots \leq i_n\}$ be an ordered subset of $\{1, \dots, 2n\}$ and let $2 \leq t \leq n$ be such that $i_t \geq n+1$ but $i_{t-1} < n+1$; that is, $I \cap \{n+1, \dots, 2n\} = \{i_t, \dots, i_n\}$.

We will use induction on t .

If $t = 2$, then $\{I\} \in A$. Suppose that the result is true for $t < k$ and consider $[I] = [i_1 \dots i_n]$ with $I \cap \{n+1, \dots, 2n\} = \{i_k, \dots, i_n\}$. We use the generalised Quantum Plücker relations (Theorem 2.3.7) to rewrite the product $[n+1 \dots 2n] [i_1 \dots i_n]$.

Let $K = \{i_1, n+1, \dots, 2n\}$, $J_1 = \emptyset$ and $J_2 = \{i_2, \dots, i_n\}$. Then

$$\sum_{K' \sqcup K'' = K} (-q)^{\ell(K':K'') + \ell(K'':J_2)} [K'] [K'' \sqcup J_2] = 0$$

where either

$$K' = \{n+1 \dots 2n\} \text{ and } K'' = \{i_1\},$$

or

$$K' = \{n+1 \dots \widehat{l} \dots 2n\} \text{ and } K'' = \{l\}$$

where $l \notin \{i_2, \dots, i_n\}$. Let $S = \{n+1, \dots, 2n\} \setminus \{i_2, \dots, i_n\}$. Then

$$(-q)^n [n+1 \dots 2n] [i_1 \dots i_n] = - \sum_{l \in S} (-q)^{\bullet} [i_1 \ n+1 \dots 2n] [l \ i_2 \dots i_n]$$

where $\bullet = \ell(K' : K'') + \ell(K'' : J_2)$. Multiplying through by $[n+1 \dots 2n]^{-2}$ gives

$$(-q)^n \{i_1 \dots i_n\} = \sum_{l \in S} (-q)^{\bullet+1} \{i_1 \ n+1 \dots 2n\} \{l \ i_2 \dots i_n\}.$$

Now $|\{l, i_2, \dots, i_n\} \cap \{n+1, \dots, 2n\}| = n - k + 2$ and so, by the induction hypothesis, $\{l \ i_2 \dots i_n\} \in A$. Clearly $\{i_1 \ n+1 \dots 2n\} \in A$, therefore $\{i_1 \dots i_n\} \in A$ and

$$A = S_0.$$

□

Theorem 3.2.6 is the result we have been aiming for: the dehomogenisation of the $n \times 2n$ quantum Grassmannian at the ‘rightmost’ minor is isomorphic to the k -algebra of $n \times n$ quantum matrices. The proof of the injectivity of the map ρ illustrates the usefulness of Gelfand-Kirillov dimension.

Theorem 3.2.6. *The map*

$$\rho : \mathcal{O}_q(M_n) \rightarrow A = S_0$$

is an isomorphism; that is,

$$\text{Dhom}(G_q(n, 2n), [n+1 \dots 2n]) \cong \mathcal{O}_q(M_n).$$

Proof. We already know that ρ is surjective. By using Gelfand-Kirillov dimension we will show that $\ker \rho = 0$.

Note that from Lemma 2.1.4 and Theorem 2.3.14, we have that

$$\text{GKdim}(\mathcal{O}_q(M_n)) = n^2 \text{ and } \text{GKdim}(G_q(n, 2n)) = n^2 + 1.$$

Suppose that $\ker \rho \neq 0$. Then, since $\mathcal{O}_q(M_n)$ is a domain,

$$\begin{aligned} \text{GKdim}(S_0) &= \text{GKdim}(\mathcal{O}_q(M_n)/\text{Ker} \rho) \\ &\leq \text{GKdim}(\mathcal{O}_q(M_n)) - 1 \\ &= n^2 - 1. \end{aligned}$$

However, we also have from Proposition 3.1.5 that

$$\text{GKdim}(S_0) = \text{GKdim}(G_q(n, 2n)) - 1 = n^2,$$

which is a contradiction. Therefore $\ker \rho = 0$ and ρ is an isomorphism. \square

The map ρ would perhaps be of more use if we knew what it did to a general minor in $\mathcal{O}_q(M_n)$.

Lemma 3.2.7. *Let $[j_t \dots j_1 | i_1 \dots i_t] \in \mathcal{O}_q(M_n)$ and let $s_l = 2n + 1 - j_l$ for $l = 1, \dots, t$. Then*

$$\rho([j_t \dots j_1 | i_1 \dots i_t]) = \{i_1 \dots i_t \ n + 1 \dots \widehat{s_t} \dots \widehat{s_1} \dots 2n\}.$$

Proof. We use induction on t , the size of the minor. If $t = 1$ then the result is true by definition. Suppose the result is true for $t < k - 1$ and consider the minor

$$[i_1 \dots i_n] = [i_1 \dots i_{k-1} \ n + 1 \dots \widehat{s_{k-1}} \dots \widehat{s_1} \dots 2n]$$

We will show that

$$\rho^{-1}(\{i_1 \dots i_n\}) = [j_{k-1} \dots j_1 \mid i_1 \dots i_{k-1}].$$

Consider the product $[n+1 \dots 2n][i_1 \dots i_n]$ in $G_q(n, 2n)$. By the proof of Lemma 3.2.5:

$$(-q)^n \{i_1 \dots i_n\} = \sum_{l=1}^{k-1} (-q)^{\bullet+1} \{i_1 \ n+1 \dots \widehat{s}_l \dots 2n\} \{l \ i_2 \dots i_n\}$$

where

$$\begin{aligned} \bullet &= \ell(\{n+1, \dots, \widehat{s}_l \dots 2n\} : \{s_l\}) + \ell(\{s_l\} : \{i_2, \dots, i_n\}) \\ &= n + l - 2. \end{aligned}$$

Therefore

$$\begin{aligned} \rho^{-1}(\{i_1 \dots i_n\}) &= \sum_{l=1}^{k-1} (-q)^{l-1} \rho^{-1}(\{i_1 \ n+1 \dots \widehat{s}_l \dots 2n\}) \rho^{-1}(\{s_l \ i_2 \dots i_n\}) \\ &= \sum_{l=1}^{k-1} (-q)^{l-1} X_{j_l, i_1} [j_{k-1} \dots \widehat{j}_l \dots j_1 \mid i_2 \dots i_{k-1}] \\ &= [j_{k-1} \dots j_1 \mid i_1 \dots i_{k-1}], \end{aligned}$$

by Lemma 3.2.2. □

The reader should note that all the results appearing in this section can be applied to the more general setting of $G_q(m, n)$. We have chosen to restrict to $G_q(n, 2n)$ to make the calculations involved easier to follow. The more general result is given in Theorem 3.2.8.

Theorem 3.2.8. *Let $1 \leq m \leq n$. Then*

$$\text{Dhom}(G_q(m, n), [n-m+1 \dots n]) \cong \mathcal{O}_q(M_{m, (n-m)}).$$

□

We concentrate on the following example in Section 4.2.

Example 3.2.9. Dehomogenisation of $G_q(2, 4)$

Let $S = G_q(2, 4)_{[34]}$. Then $\text{Dhom}(G_q(2, 4), [34]) \cong S_0$ and S_0 is generated by the elements

$$[12][34]^{-1}, \quad [13][34]^{-1}, \quad [14][34]^{-1}, \quad [23][34]^{-1}, \quad [24][34]^{-1}.$$

Recall that $\{ij\} = [ij] [34]^{-1}$. From Example 2.2.2 we can calculate the following commutation relations:

$$\{13\}\{23\} = q\{23\}\{13\}; \quad \{13\}\{14\} = q\{14\}\{13\};$$

$$\{13\}\{24\} = \{24\}\{13\} + (q - q^{-1}) \{23\}\{14\};$$

$$\{14\}\{23\} = \{23\}\{14\}; \quad \{14\}\{24\} = q\{24\}\{14\}$$

and from the Quantum Plücker relation;

$$\{12\} = \{13\}\{24\} - q\{23\}\{14\}.$$

We can immediately see the correspondence (or we can use ρ to find the correspondence):

$$\begin{array}{ccc} \mathcal{O}_q(M_2) & \longleftrightarrow & S_0 \\ a & \longleftrightarrow & \{13\} \\ b & \longleftrightarrow & \{23\} \\ c & \longleftrightarrow & \{14\} \\ d & \longleftrightarrow & \{24\} \\ D & \longleftrightarrow & \{12\}, \end{array}$$

and from Theorem 3.2.6

$$\text{Dhom}(G_q(2, 4), [34]) \cong \mathcal{O}_q(M_2).$$

In Section 4.2 we not only look at the dehomogenisation of $G_q(2, 4)$ at the minor $[34]$, we study a sequential dehomogenisation and factorisation of the algebra in order to describe its graded prime spectrum. As we will see, the motivation for employing this technique comes from the classical cell decomposition of a projective space.

Chapter 4

The 2×4 Quantum Grassmannian

In this chapter we turn our attention to the study of the simplest, while instructive, example of a quantum Grassmannian, $G_q(2, 4)$. First we demonstrate how the 2×4 quantum Grassmannian can be presented as a factor ring of an iterated Ore extension. This result not only establishes many basic ring theoretic properties for $G_q(2, 4)$, it also obtains for us some homological properties. In Theorem 4.1.2 we observe that $G_q(2, 4)$ is Auslander Gorenstein and Cohen-Macaulay. We also make the conjecture that this is true for the general quantum Grassmannian, $G_q(m, n)$.

The remainder of the chapter is devoted to finding the graded prime spectrum of $G_q(2, 4)$. Recall that the dehomogenisation of $G_q(2, 4)$ at the minor [34] is isomorphic to the coordinate ring of 2×2 quantum matrices. The prime spectrum of $\mathcal{O}_q(M_2)$ has been completely described, and so we use the correspondence found in Section 3.1 to obtain those graded primes in $G_q(2, 4)$ not containing [34]. Then, by mimicking the cell decomposition of a projective space, we describe how, by a sequential dehomogenisation and factorisation, we can completely describe the graded prime spectrum of $G_q(2, 4)$.

4.1 $G_q(2, 4)$ as the factor ring of an iterated Ore extension

Recall the commutation relations for $G_q(2, 4)$:

$$[12][13] = q[13][12], \quad [12][14] = q[14][12], \quad [12][23] = q[23][12],$$

$$[12][24] = q[24][12], \quad [12][34] = q^2[34][12], \quad [13][14] = q[14][13],$$

$$[13][23] = q[23][13], \quad [13][24] = [24][13] + (q - q^{-1})[14][23],$$

$$[13][34] = q[34][13], \quad [14][23] = [23][14], \quad [14][24] = q[24][14],$$

$$[14][34] = q[34][14], \quad [23][24] = q[24][23], \quad [23][34] = q[34][23],$$

$$[24][34] = q[34][24].$$

Following these relations, we construct an iterated Ore extension R of the field k in indeterminates $\{ij\}$ ($1 \leq i < j \leq 4$) (with the obvious intention that $\{ij\}$ will correspond to $[ij]$ in $G_q(2, 4)$). First construct the polynomial ring over k in the indeterminate $\{12\}$:

$$T_1 := k[\{12\}].$$

Next we add $\{13\}$. Define $\sigma_1 : T_1 \rightarrow T_1$ by

$$\begin{aligned} \sigma_1 : T_1 &\longrightarrow T_1 \\ \{12\} &\mapsto q^{-1}\{12\}. \end{aligned}$$

Then

$$T_2 := T_1[\{13\}; \sigma_1].$$

We now add $\{14\}$. Define

$$\begin{aligned} \sigma_2 : T_2 &\longrightarrow T_2 \\ \{12\} &\mapsto q^{-1}\{12\} \\ \{13\} &\mapsto q^{-1}\{13\}. \end{aligned}$$

Then

$$T_3 := T_2[\{14\}; \sigma_2].$$

Next we will add $[23]$ by defining

$$\begin{aligned}\sigma_3 : T_3 &\longrightarrow T_3 \\ \{12\} &\mapsto q^{-1}\{12\} \\ \{13\} &\mapsto q^{-1}\{13\} \\ \{14\} &\mapsto \{14\}\end{aligned}$$

and

$$T_4 := T_3[\{23\}; \sigma_3].$$

Now we add $\{24\}$. Notice that $[13][24] = [24][13] + (q - q^{-1})[14][23]$ and so we will need to define an automorphism σ_4 and a σ_4 -derivation on T_4 . Let

$$\begin{aligned}\sigma_4 : T_4 &\longrightarrow T_4 \\ \{12\} &\mapsto q^{-1}\{12\} \\ \{13\} &\mapsto \{13\} \\ \{14\} &\mapsto q^{-1}\{14\} \\ \{23\} &\mapsto q^{-1}\{23\}\end{aligned}$$

and

$$\begin{aligned}\delta_4 : T_4 &\longrightarrow T_4 \\ \{12\} &\mapsto 0 \\ \{13\} &\mapsto (q^{-1} - q)\{14\}\{23\} \\ \{14\} &\mapsto 0 \\ \{23\} &\mapsto 0.\end{aligned}$$

Then

$$T_5 := T_4[\{24\}; \sigma_4, \delta_4].$$

Finally we must add $\{34\}$. Define

$$\begin{aligned}\sigma_5 : T_5 &\longrightarrow T_5 \\ \{12\} &\mapsto q^{-2}\{12\} \\ \{13\} &\mapsto q^{-1}\{13\} \\ \{14\} &\mapsto q^{-1}\{14\} \\ \{23\} &\mapsto q^{-1}\{23\} \\ \{24\} &\mapsto q^{-1}\{24\}\end{aligned}$$

and

$$R := T_5[\{34\}; \sigma_5].$$

By construction, the elements of R satisfy the commutation relations for $G_q(2, 4)$ and we have a map

$$\begin{aligned} \theta &: R \rightarrow G_q(2, 4) \\ \{ij\} &\mapsto [ij]. \end{aligned}$$

Recall the Quantum Plücker relation for $G_q(2, 4)$:

$$[12][34] - q[13][24] + q^2[14][23] = 0.$$

Thus, if $p = \{12\}\{34\} - q\{13\}\{24\} + q^2\{14\}\{23\}$, then $\theta(p) = 0$. Thus $p \in \ker(\theta)$. In fact we will show that $\ker(\theta)$ is generated by p . Note that an easy calculation establishes that p is normal in R , so that $pR = Rp$. Let $\bar{R} := R/Rp$. Then we have an induced map $\bar{\theta}$ from \bar{R} to $G_q(2, 4)$:

$$\begin{aligned} \bar{\theta} &: \bar{R} \rightarrow G_q(2, 4) \\ \{\bar{ij}\} &\mapsto [ij], \end{aligned}$$

where $\{\bar{ij}\}$ is the image of $\{ij\}$ in \bar{R} .

Now, R is an \mathbb{N} -graded domain, $R = k \oplus R_1 \oplus R_2 \oplus \dots$, with $\{ij\}$ defined to have degree 1. Then p has degree 2 and

$$Rp = 0 \oplus 0 \oplus kp \oplus R_1p \oplus \dots$$

and

$$\bar{R} = k \oplus R_1 \oplus \bar{R}_2 \oplus \bar{R}_3 \oplus \dots$$

where $\bar{R}_n = R_n/R_{n-2}p$ for $n \geq 2$.

We also have that $G_q(2, 4)$ is a graded algebra with $[ij]$ defined to have degree 1:

$$G_q(2, 4) = k \oplus G_1 \oplus G_2 \oplus \dots$$

Now, if we can show that $\dim(R_1) = \dim(G_1)$ and $\dim(\bar{R}_n) = \dim(G_n)$ for all $n \geq 2$, then we are done.

Consider G_n . Then Theorem 2.3.9 implies that the elements of the form

$$[12]^{a_1} [13]^{a_2} [14]^{a_3} [24]^{a_4} [34]^{a_5} \text{ and } [12]^{b_1} [13]^{b_2} [23]^{b_3} [24]^{b_4} [34]^{b_5},$$

where $\Sigma a_i = n = \Sigma b_i$, form a basis for G_n . Thus

$$\dim G_n = 2 \binom{n+4}{4} - \binom{n+3}{3}.$$

Now consider R_n . The elements $\{12\}^{c_1}\{13\}^{c_2}\{14\}^{c_3}\{23\}^{c_4}\{24\}^{c_5}\{34\}^{c_6}$, where $\Sigma c_i = n$, form a basis for R_n . So

$$\dim(R_n) = \binom{n+5}{5}.$$

Since k is a domain, R is a domain. Thus

$$\begin{aligned} \dim(\overline{R}_n) &= \dim(R_n) - \dim(R_{n-2}) \\ &= \binom{n+5}{5} - \binom{n+3}{5}. \end{aligned}$$

Recall that $\binom{n}{k-1} + \binom{n}{k} = \binom{n+1}{k}$. Then

$$\begin{aligned} \dim(G_n) &= \binom{n+4}{4} + \left(\binom{n+4}{4} - \binom{n+3}{3} \right) \\ &= \binom{n+4}{4} + \binom{n+3}{4} \\ &= \binom{n+5}{5} - \left(\binom{n+4}{5} - \binom{n+3}{4} \right) \\ &= \binom{n+5}{5} - \binom{n+3}{5} \\ &= \dim(\overline{R}_n). \end{aligned}$$

Thus we have the required result.

Proposition 4.1.1. *The 2×4 quantum Grassmannian, $G_q(2, 4)$ is isomorphic to a factor ring of an iterated Ore extension R . In fact*

$$G_q(2, 4) \cong R/Rp$$

where $p \in R_2$ is a normal non zero divisor in R . □

Theorem 4.1.2. *The 2×4 quantum Grassmannian is Auslander Gorenstein and Cohen Macaulay.*

Proof. This follows from Theorems 1.5.11 and 1.5.12. □

Note. Though we can always write $G_q(m, n)$ as a factor ring of an iterated Ore extension, it is not at all easy to write down “minimal relations” in such a way that we can make use of Theorems 1.5.11 and 1.5.12. However, we make the following conjecture.

Conjecture 4.1.3. *$G_q(m, n)$ is Auslander Gorenstein and Cohen Macaulay.* □

4.2 The ‘Cell Decomposition’ of $G_q(2, 4)$

In the remainder of the chapter we aim to describe the cell decomposition (and therefore the graded prime spectrum) of $G_q(2, 4)$ in the case that the field k is algebraically closed and $0 \neq q \in k$ is not a root of unity.

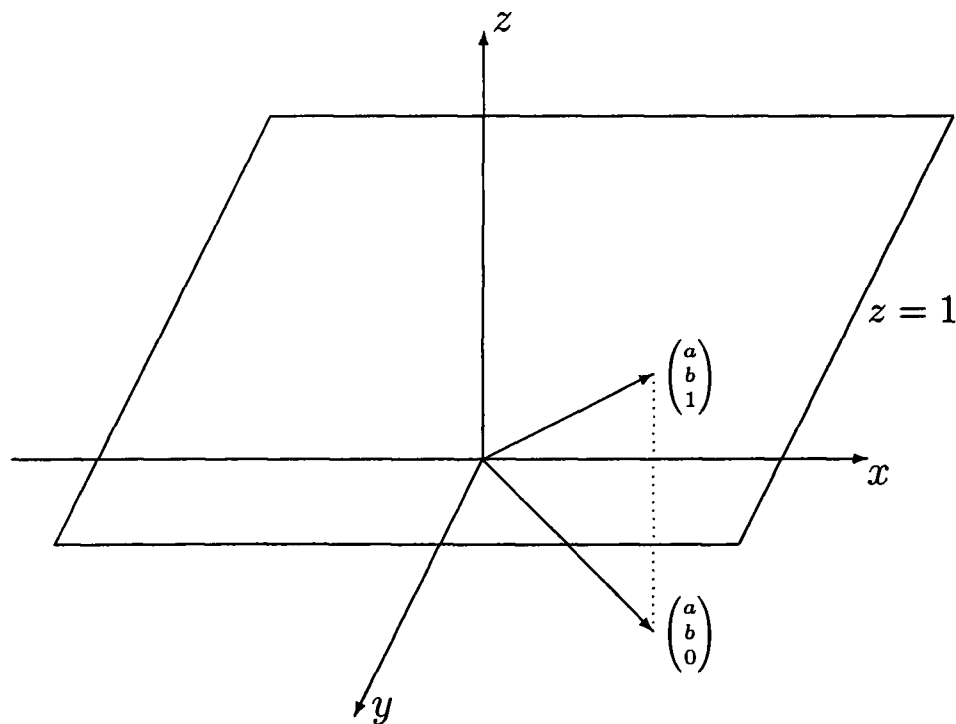
From Example 3.2.9 we have that $\text{Dhom}(G_q(2, 4), [34]) \cong \mathcal{O}_q(M_2)$. The prime spectrum of $\mathcal{O}_q(M_2)$ has been completely described by Goodearl [13] (and almost certainly others). Thus using the homeomorphism

$$\Gamma_{[34]}^{-1} : \sigma_{[34]}\text{-Spec}\mathcal{O}_q(M_2) \rightarrow \{P \in \text{GrSpec}G_q(2, 4) \mid [34] \notin P\}$$

from Section 3.1, we can hope to find those graded primes in $G_q(2, 4)$ not containing $[34]$. However, we would like to completely describe the graded prime spectrum of $G_q(2, 4)$. To obtain a method for doing this we look to the classical theory for inspiration.

The 2×4 quantum Grassmannian is a deformation of a coordinate ring of a projective manifold and a projective manifold has a cell decomposition. This cell decomposition presents the space as a disjoint union of cells; as an example, we look at the cell decomposition of projective 2-space, \mathbb{P}^2 .

Example 4.2.1. The Cell decomposition of \mathbb{P}^2 .



We can think of \mathbb{P}^2 as being $\mathbb{A}^2 \cup \{\text{points at infinity}\}$ in the following manner: the plane defined by $z = 1$ is a copy of \mathbb{A}^2 and each point at infinity is in one to one correspondence with a direction vector in the plane $z = 0$ (we should think of the projective line being ‘wrapped’ around the xy -plane at infinity).

Thus $\mathbb{P}^2 = \mathbb{A}^2 \sqcup \mathbb{P}^1$. By further decomposing $\mathbb{P}^1 = \mathbb{A}^1 \sqcup \mathbb{A}^0$, we obtain the cell decomposition

$$\mathbb{P}^2 = \mathbb{A}^2 \sqcup \mathbb{A}^1 \sqcup \mathbb{A}^0.$$

Alternatively, we can think of \mathbb{P}^2 as being lines through the origin in \mathbb{A}^3 . Then

$$\begin{aligned} \mathbb{P}^2 &= \{\text{direction vectors in } \mathbb{A}^3\} \\ &= \{[a : b : c] \mid (a, b, c) \in \mathbb{A}^3 \setminus \{0\}\}. \end{aligned}$$

So $[a : b : c] = [\alpha : \beta : \gamma]$ in \mathbb{P}^2 if and only if there exists $0 \neq \lambda \in k$ such that $(a, b, c) = \lambda(\alpha, \beta, \gamma)$. Consider a point $[a : b : c] \in \mathbb{P}^2$. Exactly one of the following is true.

(1) $c \neq 0$. Then $[a : b : c] = [u : v : 1]$ where $u = a/c$ and $v = b/c$. Then $[u : v : 1] = [\alpha : \beta : 1]$ if and only if $u = \alpha$ and $v = \beta$. Thus

$$[u : v : 1] \leftrightarrow (u, v) \in \mathbb{A}^2,$$

and therefore

$$\{[a : b : c] \in \mathbb{P}^2 \mid c \neq 0\} \cong \mathbb{A}^2.$$

(2) $c = 0$ and $b \neq 0$. Then $[a : b : c] = [w : 1 : 0]$, where $w = a/b$. Then $[w : 1 : 0] = [\alpha : 1 : 0]$ if and only if $w = \alpha$. Therefore

$$[w : 1 : 0] \leftrightarrow w \in \mathbb{A}^1$$

and

$$\{[a : b : c] \in \mathbb{P}^2 \mid c = 0, b \neq 0\} \cong \mathbb{A}^1.$$

Note how this case ties in with the geometrical discussion, since $[a : b : 0] = [\alpha : \beta : 0]$ if and only if $(a, b) = \lambda(\alpha, \beta)$ for some $0 \neq \lambda \in k$, if and only if (a, b) and (α, β) determine the same direction vector in the plane.

(3) $c = b = 0$ and $a \neq 0$. Then $[a : b : c] = [1 : 0 : 0]$ and therefore

$$\{[a : b : c] \in \mathbb{P}^2 \mid c = b = 0, a \neq 0\} \cong \mathbb{A}^0.$$

Therefore we again have the cell decomposition

$$\mathbb{P}^2 = \mathbb{A}^2 \sqcup \mathbb{A}^1 \sqcup \mathbb{A}^0.$$

We can use this cell decomposition to obtain a disjoint partition of the graded maximal ideals of the coordinate ring of \mathbb{P}^2 , $k[x, y, z]$. The projective Nullstellensatz gives a one to one correspondence between points in \mathbb{P}^2 and the relevant maximal graded ideals of $k[x, y, z]$:

$$[a : b : c] \longleftrightarrow \langle bx - ay, cx - az, cy - bz \rangle =: I.$$

Then, as before, the point $[a : b : c] \in \mathbb{P}^2$ sits inside exactly one of three cases. We work through the three cases again, this time considering the ideal I .

(1) $c \neq 0$

Suppose that $z \in I$. Then since $cy - bz \in I$, we have $y \in I$ and since $cx - az \in I$, we have $x \in I$. Therefore $\langle x, y, z \rangle \subseteq I$ and I is an irrelevant ideal. Therefore $z \notin I$ and we dehomogenise $k[x, y, z]$ at z ; that is, we factor $k[x, y, z]$ by the ideal generated by $z - 1$. Now, $I = \langle vx - uy, x - uz, y - vz \rangle = \langle x - uz, y - vz \rangle$, so the dehomogenisation of I is

$$\pi(I) = \langle x - u, y - v \rangle$$

and

$$\langle x - u, y - v \rangle \in \max\text{Spec}(k[x, y]).$$

Notice that $k[x, y]$ is the coordinate ring of \mathbb{A}^2 and that Hilbert's Nullstellensatz gives us the correspondence

$$(u, v) \longleftrightarrow \langle x - u, y - v \rangle$$

between points in \mathbb{A}^2 (and therefore points of the form $[u : v : 1]$ in \mathbb{P}^2) and the maximal ideals of $k[x, y]$.

(2) $c = 0, b \neq 0$

Then $[a : b : c] = [w : 1 : 0]$ and $I = \langle x - wy, z \rangle$, so clearly, $z \in I$. Factor $k[x, y, z]$ by z and identify I with its image in the factor ring. Since I is not an irrelevant ideal, $y \notin I$ and we dehomogenise $k[x, y, z]/\langle z \rangle$ at y . The dehomogenisation of I is

$$\pi(I) = \langle x - w \rangle \in \max\text{Spec}(k[x]).$$

Notice that $\mathcal{O}(\mathbb{A}^1)$, the coordinate ring of \mathbb{A}^1 , is equal to $k[x]$ and that the Nullstellensatz gives us a correspondence

$$w \longleftrightarrow \langle x - w \rangle$$

between points in \mathbb{A}^1 (and therefore points of the form $[w : 1 : 0]$ in \mathbb{P}^2) and the maximal ideals in $k[x]$.

(3) $c = b = 0, a \neq 0$

Then $[a : b : c] = [1 : 0 : 0]$ and $I = \langle y, z \rangle$. We factor $k[x, y, z]$ by the ideal generated by y and z and dehomogenise at x . Identify I with its image in the factor ring $k[x, y, z] / \langle y, z \rangle$. Then

$$\pi(I) = \langle 1 \rangle \in \max\text{Spec}(k).$$

Therefore the cell decomposition of \mathbb{P}^2 not only provides us with a disjoint partition of the space, we also have a disjoint partition of the relevant graded maximal ideals of the coordinate ring of the space:

$$\max\text{GrSpec}(k[x, y, x]) = A_2 \sqcup A_1 \sqcup A_0,$$

where

$$A_2 \approx \max\text{Spec}(k[x, y]),$$

$$A_1 \approx \max\text{Spec}(k[x]) \text{ and}$$

$$A_0 \approx \max\text{Spec}(k).$$

The cell decomposition of $G(2, 4)$ is well known (see for example, [11], Chapter 1, Section 5):

$$G(2, 4) = \mathbb{C}^4 \sqcup \mathbb{C}^3 \sqcup \mathbb{C}^2 \sqcup \mathbb{C}^2 \sqcup \mathbb{C}^1 \sqcup \mathbb{C}^0.$$

As in the case of projective 2-space, this decomposition will provide us with a disjoint partition of the relevant graded maximal ideals of the coordinate ring of $G(2, 4)$. In the discussion that follows we wish to give an analogous partition for the relevant graded prime ideals of $G_q(2, 4)$. The discussion in the classical case suggests that we consider sequentially dehomogenising and factorising $G_q(2, 4)$.

Before proceeding, we require some general results. Recall that if R is a noetherian ring and I is an ideal of R , then the **nil radical**, $N(I)$, of I is the intersection of all those prime ideals in R minimal over I .

Lemma 4.2.2. *Let R be a noetherian ring and let σ be an automorphism of R . If I is a σ -ideal of R , then $N(I)$ is also σ -stable.*

Proof. Let $\{Q_j\}_{j \in J}$ be the family of primes ideals of R minimal over I . If P is prime, then $\sigma(P)$ is prime and $\{\sigma(Q_j)\}_{j \in J}$ is the family of prime ideals minimal

over $\sigma(I)$. Therefore

$$\begin{aligned} N(I) = N(\sigma(I)) &= \bigcap_{j \in J} \sigma(Q_j) \\ &= \sigma\left(\bigcap_{j \in J} Q_j\right) \\ &= \sigma(N(I)). \end{aligned}$$

□

Lemma 4.2.3. *Let R be a noetherian ring and let σ be an automorphism of R . If $P \in \sigma\text{-Spec}(R)$, then P is semiprime.*

Proof. By Lemma 4.2.2, $N(P)$ is σ -stable. Also, $N(P)^m \subseteq P$ for some $m \in \mathbb{N}$, and therefore $N(P) = P$. □

Proposition 4.2.4. *Let R be a noetherian ring and $\sigma : R \rightarrow R$ be an automorphism of R . Then*

$$\sigma\text{-Spec}(R) = \{(P : \sigma) \mid P \in \text{Spec}(R)\},$$

where $(P : \sigma) = \bigcap \sigma^n(P)$. Furthermore, if P is a σ -prime ideal in R and Q is a minimal prime over P , then $(Q : \sigma) = P$.

Proof. Suppose $P \in \text{Spec}(R)$ and let $(P : \sigma) = Q \subseteq P$. Then, clearly, Q is σ -stable. Let A, B be σ -ideals of R such that $AB \subseteq Q$. Then $AB \subseteq P$ and thus, without loss of generality, $A \subseteq P$. Thus $A = \sigma^n(A) \subseteq \sigma^n(P)$ for all n and therefore $A \subseteq Q$; that is, $Q \in \sigma\text{-Spec}(R)$.

Now suppose $P \in \sigma\text{-Spec}(R)$. Then by Lemma 4.2.3, P is semiprime. Therefore

$$P = Q_1 \cap Q_2 \cap \dots \cap Q_l$$

where Q_i are the minimal primes over P (of which there are finitely many). Then

$$P = \sigma^n(P) = \sigma^n(Q_1) \cap \sigma^n(Q_2) \cap \dots \cap \sigma^n(Q_l)$$

and

$$P = \bigcap_{i=1}^l (Q_i : \sigma) \supseteq \prod_{i=1}^l (Q_i : \sigma).$$

Now, P is σ -prime and so, without loss of generality, $(Q_1 : \sigma) \subseteq P$ and since $P \subseteq Q_1$, we have $P = (Q_1 : \sigma)$, as required.

Let $Q = Q_i$ for some $1 \leq i \leq l$. Each $\sigma^j(Q)$ is a minimal prime over P , of which there are finitely many. Therefore $\sigma^n(Q) = Q$ for some n . Let $X = \{\sigma^j(Q) \mid j =$

$1, \dots, n-1\}$. If $n = l$, then $X = \{Q_1, \dots, Q_l\}$ and we are done. Otherwise, let $Y = \{Q_1, \dots, Q_l\} \setminus X = \{I_1, \dots, I_s\}$ say, and let $J := \cap I_j$. Then J is a σ -stable ideal and

$$P = J \cap (Q : \sigma) \supseteq J(Q : \sigma).$$

Thus $J \subseteq P$ or $(Q : \sigma) \subseteq P$. Suppose $J \subseteq P$. Then $I_1 \dots I_s \subseteq J \subseteq P \subseteq Q$ and $I_j \subseteq Q$ for some j . However, since Q and I_j are minimal over P , we have that $Q = I_j$, which is a contradiction. Therefore $P = (Q : \sigma)$. \square

Lemma 4.2.4 says that for any minimal primes Q_1, Q_2 over a σ -prime P , we have that $(Q_1 : \sigma) = P = (Q_2 : \sigma)$ and so $Q_1 = \sigma^i(Q_2)$ for some i . Therefore, in the proof above we always have the case that $X = \{Q_1, \dots, Q_l\}$.

The ‘Cell Decomposition’ of $G_q(2, 4)$

Let $0 \neq q \in k$ such that $q^n \neq 1$ for any n and let $G := G_q(2, 4)$. Figure 4.1 shows the partial ordering \leq_c (defined in 2.3.1) on the minors of G . Let P be a relevant

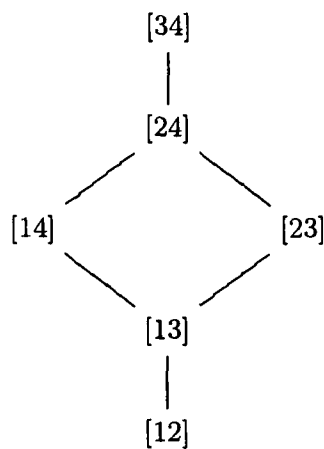


Figure 4.1: The partial ordering \leq_c on $G_q(2, 4)$

graded prime ideal of G . Then exactly one of the following cases holds:

1. $[34] \notin P$,
2. $[34] \in P, [24] \notin P$,
3. $[34], [24] \in P, [23] \notin P$,
4. $[34], [24], [23] \in P, [14] \notin P$,
5. $[34], [24], [23], [14] \in P, [13] \notin P$,
6. $[34], [24], [23], [14], [13] \in P, [12] \notin P$.

Note. In the following discussion we will use $[ij]$ to denote the minor in G and also its image in the various factor rings that appear. The way in which we are thinking of $[ij]$ should be clear from the context.

Case 1: $[34] \notin P$

We have seen that $[34]$ is a regular normal element in G and that if $S := G_{[34]}$, then

$$\text{Dhom}(G, [34]) = S_0 \cong \mathcal{O}_q(M_2).$$

Let $\sigma_{[34]} : S_0 \rightarrow S_0$ be the automorphism obtained by conjugating by $[34]$ and recall the homeomorphism

$$\Gamma_{[34]} : \{P \in \text{GrSpec}(G) \mid [34] \notin P\} \longrightarrow \sigma_{[34]}\text{-Spec}(S_0)$$

where $\Gamma_{[34]}(P) = PS \cap S_0$.

Therefore, in order to find those graded primes in G not containing $[34]$, we must identify the $\sigma_{[34]}$ -prime spectrum of $\mathcal{O}_q(M_2)$. Recall the isomorphism ρ from Example 3.2.9:

$$\begin{aligned} \rho : \mathcal{O}_q(M_2) &\longrightarrow S_0 \\ a &\mapsto [13][34]^{-1} \\ b &\mapsto [23][34]^{-1} \\ c &\mapsto [14][34]^{-1} \\ d &\mapsto [24][34]^{-1} \\ D &\mapsto [12][34]^{-1}. \end{aligned}$$

Then from the commutation relations for G we can calculate

$$\begin{aligned} \sigma_{[34]} : \mathcal{O}_q(M_2) &\longrightarrow \mathcal{O}_q(M_2) \\ a &\mapsto q^{-1}a \\ b &\mapsto q^{-1}b \\ c &\mapsto q^{-1}c \\ d &\mapsto q^{-1}d \\ D &\mapsto q^{-2}D. \end{aligned}$$

Let $A := \mathcal{O}_q(M_2)$. Goodearl [13] has completely described the prime spectrum of A in the following way. The torus $\mathcal{H} = (k^\times)^4$ acts on A via

$$(\alpha, \beta; \gamma, \delta) \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \alpha\gamma a & \alpha\delta b \\ \beta\gamma c & \beta\delta d \end{pmatrix}.$$

Under this action there are 14 \mathcal{H} -stable prime ideals which are displayed in Figure 4.2 (taken from [19]). Let $P \in \operatorname{Spec}(A)$ and define $(P : \mathcal{H}) := \bigcap_{h \in H} P^h$. Then

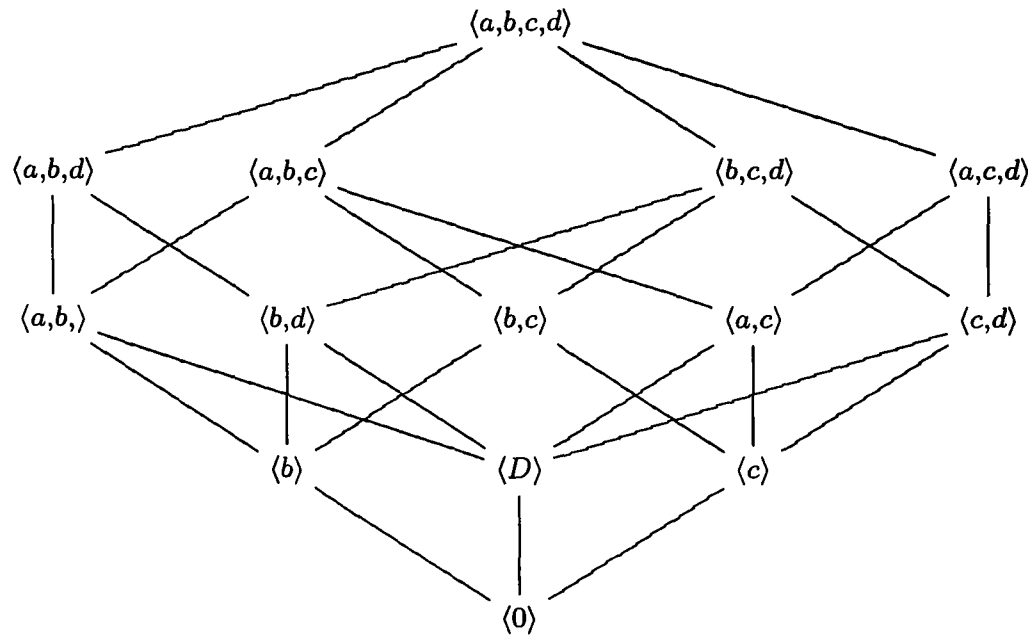


Figure 4.2: The \mathcal{H} -stable primes of $\mathcal{O}_q(M_2)$

each $(P : \mathcal{H})$ is one of the 14 \mathcal{H} -stable prime ideals in Figure 4.2 and this gives us a stratification of the prime spectrum of A . Let J be an \mathcal{H} -prime and define

$$\operatorname{Spec}_J(A) = \{P \in \operatorname{Spec}(A) \mid (P : \mathcal{H}) = J\}.$$

Then

$$\operatorname{Spec}(A) = \bigsqcup_{J \in \mathcal{H}\text{-Spec}(A)} \operatorname{Spec}_J(A).$$

Thus one aims to identify $\operatorname{Spec}_J(A)$ for each of the possibilities for J . This is possible by following the recipe given in [22] Theorem 6.6. For example, when $J = 0$ we have homeomorphisms (via extension and contraction)

$$\operatorname{Spec}_0(A) \rightarrow \operatorname{Spec}(B) \rightarrow \operatorname{Spec}(Z(B)),$$

where $B = A[b^{-1}, c^{-1}, D^{-1}]$ and $Z(B) = k[(bc^{-1})^{\pm 1}, D^{\pm 1}]$ is the centre of B . By identifying the prime spectrum of $Z(B)$ and using these homeomorphisms, we obtain

$$\operatorname{Spec}_0(A) = \{\langle 0 \rangle, \langle b - \alpha c, D - \beta \rangle, \langle f \rangle \cap A\}$$

where $0 \neq \alpha, \beta \in k$ and $f \in Z(B)$ is irreducible. Continuing in this way, we can complete the stratification of $\operatorname{Spec}(A)$. In 5 cases the stratum $\operatorname{Spec}_J(A)$ is a singleton:

$$\operatorname{Spec}_{\langle b, a \rangle}(A) = \{\langle b, a \rangle\}, \quad \operatorname{Spec}_{\langle b, d \rangle}(A) = \{\langle b, d \rangle\}, \quad \operatorname{Spec}_{\langle c, a \rangle}(A) = \{\langle c, a \rangle\},$$

$$\text{Spec}_{\langle c, d \rangle}(A) = \{\langle c, d \rangle\}, \quad \text{Spec}_{\langle d, c, b, a \rangle}(A) = \{\langle d, c, b, a \rangle\}.$$

In 7 cases, $\text{Spec}_J(A)$ is 1-dimensional:

$$\begin{aligned} \text{Spec}_{\langle b \rangle}(A) &= \{\langle b \rangle, \langle b, ad - \alpha \rangle\}, \\ \text{Spec}_{\langle c \rangle}(A) &= \{\langle c \rangle, \langle c, ad - \alpha \rangle\}, \\ \text{Spec}_{\langle D \rangle}(A) &= \{\langle D \rangle, \langle D, b - \alpha c \rangle\}, \\ \text{Spec}_{\langle b, a, d \rangle}(A) &= \{\langle b, a, d \rangle, \langle b, a, d, c - \alpha \rangle\}, \\ \text{Spec}_{\langle c, a, d \rangle}(A) &= \{\langle c, a, d \rangle, \langle c, a, d, b - \alpha \rangle\}, \\ \text{Spec}_{\langle b, c, a \rangle}(A) &= \{\langle b, c, a \rangle, \langle b, c, a, d - \alpha \rangle\}, \\ \text{Spec}_{\langle b, c, d \rangle}(A) &= \{\langle b, c, d \rangle, \langle b, c, d, a - \alpha \rangle\} \end{aligned}$$

where in each case $0 \neq \alpha \in k$. The remaining 2 cases are 2-dimensional. As we have seen

$$\text{Spec}_0(A) = \{\langle 0 \rangle, \langle b - \alpha c, D - \beta \rangle, \langle f \rangle \cap A\}$$

where $0 \neq \alpha, \beta \in k$ and $f \in k[(bc^{-1})^{\pm 1}, D^{\pm 1}]$ is irreducible. Finally

$$\text{Spec}_{\langle b, c \rangle}(A) = \{\langle b, c \rangle, \langle b, c, a - \alpha, d - \beta \rangle, \langle b, c, g \rangle \cap A\}$$

where $0 \neq \alpha, \beta \in k$ and $g \in k[a^{\pm 1}, d^{\pm 1}]$ is irreducible.

By Lemma 4.2.4

$$\sigma_{[34]}-\text{Spec}(A) = \{(P : \sigma_{[34]}) \mid P \in \text{Spec}(A)\}$$

and so we must find $(P : \sigma_{[34]})$ for each of the primes above. Note that the action of $\sigma_{[34]}$ on A corresponds to the action of $(q^{-1}, q^{-1}, 1, 1) \in \mathcal{H}$. Thus the \mathcal{H} -stable primes are $\sigma_{[34]}$ -primes. For many of the remaining primes identifying $(P : \sigma_{[34]})$ proves to be relatively straightforward. The only problematic cases arise when $P = \langle f \rangle \cap A$ and $P = \langle b, c, g \rangle \cap A$. Here, we give a straightforward example of finding $(P : \sigma_{[34]})$ when $P = \langle b - \alpha c, D - \beta \rangle$ and then use this to find the $\sigma_{[34]}$ -prime, $(\langle f \rangle \cap A : \sigma_{[34]})$.

Let $P = \langle b - \alpha c, D - \beta \rangle$. Then

$$\begin{aligned} (P : \sigma_{[34]}) &= \cap \sigma_{[34]}^n(P) \\ &= \cap \langle b - \alpha c, D - q^{2n}\beta \rangle \\ &= \langle b - \alpha c \rangle. \end{aligned}$$

The final equality here stems from each of the ideals $\langle b - \alpha c, D - q^{2n}\beta \rangle$ being distinct and $A/\langle b - \alpha c \rangle$ having dimension 1.

Now let $P = \langle f \rangle \cap A = fB \cap A$ where $B = A[b^{-1}, c^{-1}, D^{-1}]$ and f is irreducible in $Z(B) = k[(bc^{-1})^{\pm 1}, D^{\pm 1}]$. The homeomorphisms mentioned earlier give the following correspondence:

$$\begin{aligned} \text{Spec}_0(A) &\rightarrow \text{Spec}(B) \rightarrow \text{Spec}(Z(B)) \\ fB \cap A &\mapsto fB \mapsto fZ(B). \end{aligned}$$

Then $fZ(B)$ is contained in a maximal ideal of $Z(B)$, so

$$fZ(B) \subseteq \langle bc^{-1} - \alpha, D - \beta \rangle$$

for some $0 \neq \alpha, \beta \in k$. Therefore $P = fB \cap A \subseteq \langle b - \alpha c, D - \beta \rangle$ and

$$(P : \sigma_{[34]}) \subseteq \cap \sigma_{[34]}^n(\langle b - \alpha c, D - \beta \rangle) = \langle b - \alpha c \rangle.$$

Suppose that $(P : \sigma_{[34]}) \neq 0$. Then since $\langle b - \alpha c \rangle$ is a prime of height 1, we must have that $\langle b - \alpha c \rangle$ is minimal over $(P : \sigma_{[34]})$. Therefore, by Lemma 4.2.4,

$$(P : \sigma_{[34]}) = (\langle b - \alpha c \rangle : \sigma_{[34]}) = \langle b - \alpha c \rangle.$$

The remaining $\sigma_{[34]}$ -primes are found in the same manner and we are now in a position to write down the $\sigma_{[34]}$ -prime spectrum of A :

$$\begin{aligned} \sigma_{[34]}\text{-Spec}(A) = & \{ \langle 0 \rangle, \langle b \rangle, \langle c \rangle, \langle D \rangle, \langle b - \alpha c \rangle, \langle b - \alpha c, D \rangle, \\ & \langle b, a \rangle, \langle b, d \rangle, \langle c, a \rangle, \langle c, d \rangle, \langle b, c \rangle, \langle b, a, d \rangle, \\ & \langle b, c, a - \alpha d \rangle, \langle c, a, d \rangle, \langle b, c, a \rangle, \langle b, c, d \rangle, \\ & \langle b, c, d, a \rangle \} \end{aligned}$$

where $0 \neq \alpha \in k$. Let $\widetilde{[ij]} := [ij][34]^{-1}$. Then Figure 4.3 shows the poset of $\sigma_{[34]}$ -primes in S_0 (where the dotted line indicates inclusion if and only if $\alpha = \beta$).

We should trace these ideals back to G using $\Gamma_{[34]}^{-1}$. Let I be a graded ideal of G . Recall that if G/I is $[34]$ -torsion free and $\Gamma_{[34]}(I) = J \in \sigma_{[34]}\text{-Spec}S_0$; then $\Gamma_{[34]}^{-1}(J) = I$. Consider the ideal $\langle \widetilde{[13]}, \widetilde{[23]} \rangle = \langle \widetilde{[12]}, \widetilde{[13]}, \widetilde{[23]} \rangle \in \sigma_{[34]}\text{-Spec}S_0$. Then $\Gamma_{[34]}(\langle [12], [13], [23] \rangle) = \langle \widetilde{[12]}, \widetilde{[13]}, \widetilde{[23]} \rangle$. We will show that the factor ring $G/\langle [12], [13], [23] \rangle$ is $[34]$ -torsion free. Recall that a basis for G is given by elements of the form

$$[12]^{a_1} [13]^{a_2} [14]^{a_3} [24]^{a_4} [34]^{a_5} \quad \text{and} \quad [12]^{b_1} [13]^{b_2} [23]^{b_3} [24]^{b_4} [34]^{b_5}$$

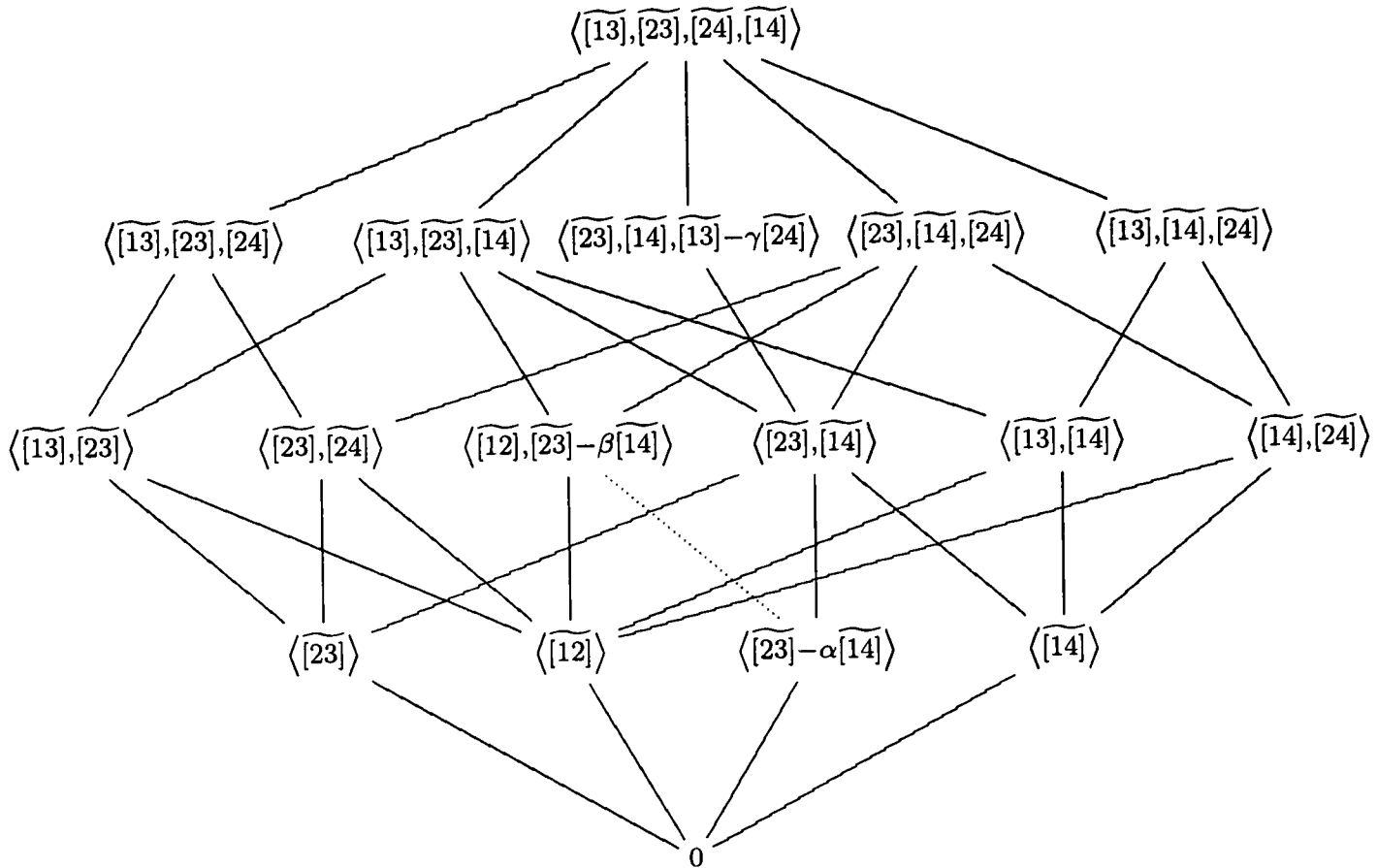


Figure 4.3: The poset $\sigma_{[34]}$ -Spectrum of S_0 ($\alpha, \beta, \gamma \in k^\times$)

and we can represent such a product by a tableau $[T]$ and refer to it as a preferred product. Let $I = \langle [12], [13], [23] \rangle = [12]G + [13]G + [23]G$. We claim that a k -basis for I is given by those preferred products containing at least one of $[12]$, $[13]$ or $[23]$ non-trivially. That these products are linearly independent is clear. Let V be the space spanned by such products. Then $V \subseteq I$. Suppose that there exists $y = \sum \alpha_i [T_i] \in I \setminus V$ with $\alpha_i \neq 0$ where $[T_i]$ is a preferred product and, without loss of generality, $[T_i]$ does not contain $[12]$, $[13]$ or $[23]$ non-trivially, for any i . Now, $y \in I$ and so

$$y = [12]g + [13]h + [23]j$$

where $g = \sum \beta_i [S_i]$, $h = \sum \delta_i [U_i]$ and $j = \sum \gamma_i [V_i]$ for some preferred products $[S_i]$, $[U_i]$ and $[V_i]$. Then

$$[12]g = \sum \beta_i [12][S_i]$$

and $[12][S_i]$ is clearly a preferred product containing $[12]$ non-trivially. Also,

$$[13]h = \sum \delta_i [13][U_i] = \sum \delta'_i [U'_i]$$

where $[U'_i]$ is a preferred product containing $[13]$ non-trivially. Finally consider $[23][V_i]$. If the preferred product $[V_i]$ does not contain $[14]$, then $\gamma_i [23][V_i] =$

$\gamma'_i [V'_i]$ where $[V'_i]$ is a preferred product containing $[23]$ non-trivially, since $[23]$ q -commutes with all the other generators. If $[V_i]$ does contain $[14]$, then we must rewrite any product $[23][14]$ appearing using the Quantum Plücker relation and we can rewrite $\gamma_i [23][V_i]$ as a sum of preferred products $\gamma'_i [V'_i]$ where $[V'_i]$ contains either $[12]$ or $[13]$ non-trivially. Therefore

$$y = \sum \omega_i [W_i]$$

where each $[W_i]$ is a preferred product containing at least one of $[12], [13], [23]$ non-trivially. Then

$$\sum \omega_i [W_i] = \sum \alpha_i [T_i]$$

which is a linear equation between elements of a basis. Therefore each $[T_j]$ is equal to some $[W_i]$ and we have a contradiction. Therefore $V = I$.

Now consider the factor ring $\overline{G} = G/I$. The images of those preferred products not containing $[12], [13]$ or $[23]$ span \overline{G} . Suppose that they are not linearly independent. Then there exists preferred products $[T_i]$ not containing $[12], [13]$ or $[23]$ non-trivially such that

$$\sum \alpha_i [T_i] \in [12]G + [13]G + [23]G.$$

By the same argument as above we obtain a contradiction. Suppose that $[34]$ is a zero divisor in \overline{G} . So there exists $g \in G \setminus I$ such that $g[34] \in I$ and without loss of generality, g is a sum of preferred products not containing $[12], [13]$ or $[23]$; that is, $g = \sum \alpha_i [T_i]$. Then $g[34] = \sum \alpha_i [T_i][34]$. However, we also have that

$$g[34] = \sum \beta_i [S_i]$$

where $[S_i]$ is a preferred product containing $[12], [13]$ or $[23]$ non-trivially. Thus, once again we obtain a contradiction. Therefore G/I is $[34]$ -torsion free and

$$\Gamma_{[34]}^{-1} \left(\left\langle \widetilde{[13]}, \widetilde{[23]} \right\rangle \right) = \langle [12], [13], [23] \rangle.$$

A similar argument can be repeated for all but six of the ideals in $\sigma_{[34]} \text{-Spec} S_0$. The poset of graded primes in G not containing $[34]$ is displayed in Figure 4.4 (where the dotted line indicates inclusion if and only if $\alpha = \beta$).

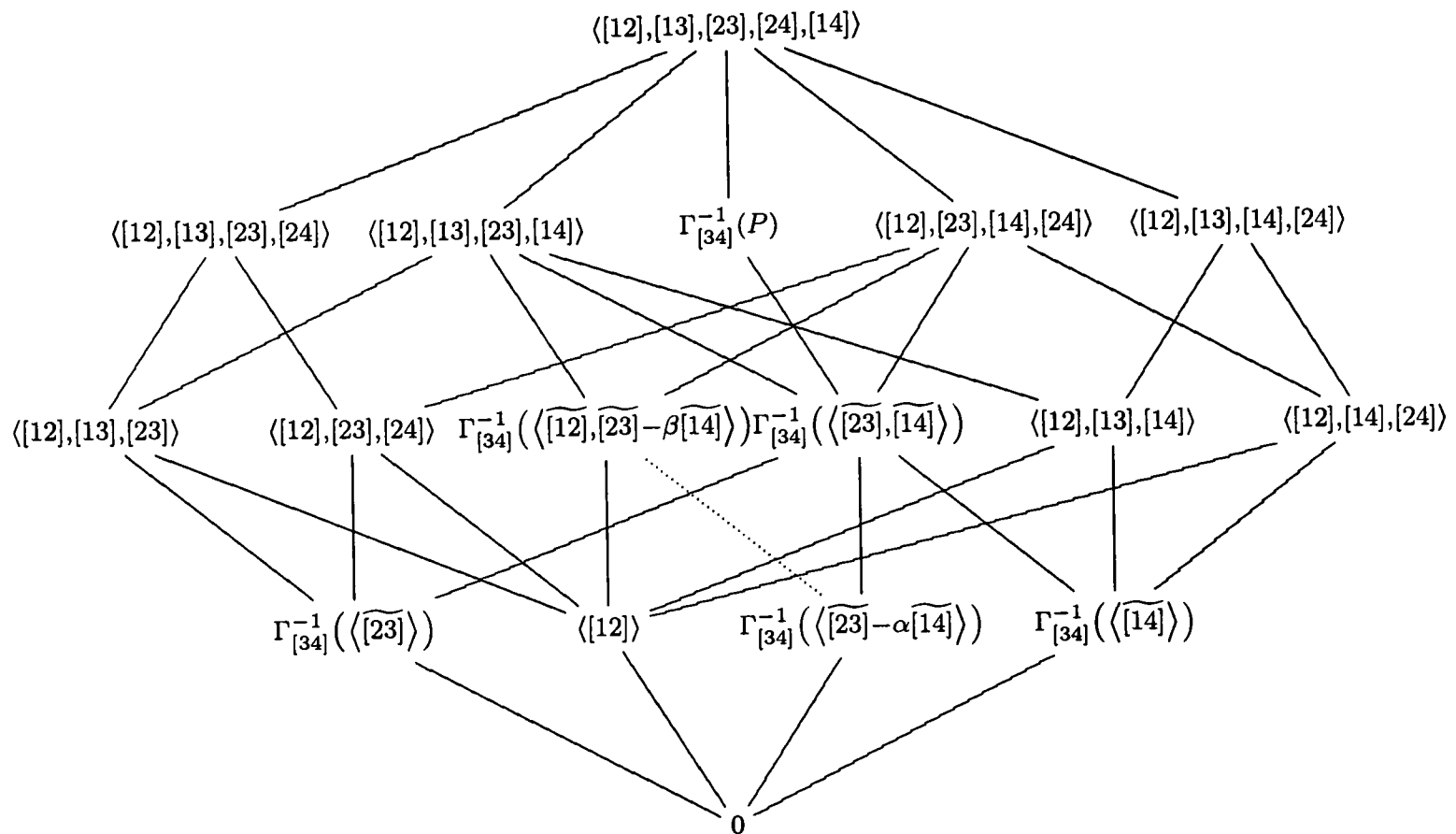


Figure 4.4: Poset of graded primes in G not containing $[34]$, where $\alpha, \beta, \gamma \in k^\times$ and $P := \langle \widetilde{[14]}, \widetilde{[23]}, \widetilde{[13]} - \gamma \widetilde{[24]} \rangle$

Case 2: $[34] \in P$, $[24] \notin P$

As suggested by the classical theory, we proceed by factoring G by the ideal generated by $[34]$ and identifying P with its image in this factor ring. A brief inspection of the commutation relations and the Quantum Plücker relation for G establishes that $[24]$ is normal modulo the ideal generated by the minor $[34]$. We also require that $[24]$ is regular modulo the ideal generated by $[34]$. Note that $[T]$ is a preferred product if and only if $[T][34]$ is. Let $I := G[34] = [34]G$. Then a k -basis for I is given by the preferred products which contain $[34]$ non-trivially. That these preferred products are linearly independent is clear. Let V be the space spanned by such products. Then certainly $V \subseteq I$. Suppose that there exists $y = \sum \alpha_i [T_i] \in I \setminus V$ with $\alpha_i \neq 0$ and, without loss of generality, such that $[34]$ does not appear in $[T_i]$ for any i . Then since $y \in I$, we have $y = a[34]$ for some $a \in G$. We can write a as a k -linear combination of preferred products, $a = \sum \beta_j [S_j]$. Then

$$y = \sum \alpha_i [T_i] = \sum \beta_j [S_j] [34]$$

which is a linear equation between elements of a basis and so $[T_i] = [S_j][34]$, contradicting the assumption that $[34]$ does not appear in $[T_i]$. Therefore $V = I$.

Consider the factor ring $G_1 := G/I$. The images of those preferred products not involving $[34]$ form a k -basis for G_1 . These images obviously span G_1 . Suppose they are not linearly independent. Then there exist preferred products $[T_i]$ not containing $[34]$ such that $\sum \alpha_i [T_i] \in I$ with $\alpha_i \neq 0$. By the same argument as above, this yields a contradiction.

Now suppose that the image of $[24]$ is a zero divisor in G_1 ; that is, there exists $a \in G \setminus I$ such that $a[24] \in I$. Then, without loss of generality, we can write a as a k -linear combination of preferred products not containing $[34]$. So

$$a = \sum \alpha_i [T_i].$$

Then

$$a[24] = \sum \alpha_i [T_i][24]$$

and note that $[T_i][24]$ is a preferred product, since $[T_i]$ does not involve $[34]$. However, we must also have that

$$a[24] = \sum \beta_j [S_j]$$

where the $[S_j]$ are preferred products involving $[34]$ non-trivially, thus obtaining a contradiction in the now usual way.

Therefore $[24]$ is a regular normal element in $G_1 = G/\langle [34] \rangle$ and we may invert $[24]$ in G_1 . Let T be the localisation of G_1 at $[24]$; that is,

$$T := (G_1)_{[24]}.$$

Then T is \mathbb{Z} -graded and $T_0 = \text{Dhom}(G_1, [24])$ is generated by

$$[12][24]^{-1}, [13][24]^{-1}, [14][24]^{-1} \text{ and } [23][24]^{-1}.$$

From the Quantum Plücker relation modulo $[34]$ we have that $q^2[14][23] = q[13][24]$ in G_1 . Therefore in T ,

$$[13][24]^{-1} = [14][24]^{-1}[23][24]^{-1}.$$

Thus a generating set for $\text{Dhom}(G_1, [24])$ is in fact

$$[12][24]^{-1}, [14][24]^{-1} \text{ and } [23][24]^{-1}.$$

Using the commutation relations for G modulo $[34]$, we obtain the commutation relations:

$$[12][24]^{-1}[14][24]^{-1} = q[14][24]^{-1}[12][24]^{-1},$$

$$[12][24]^{-1}[23][24]^{-1} = q[23][24]^{-1}[12][24]^{-1},$$

$$[14][24]^{-1}[23][24]^{-1} = [23][24]^{-1}[14][24]^{-1}.$$

Let R be the k -algebra generated by the three indeterminates X, Y, Z subject to the relations

$$XY = qYX, \quad XZ = qZX, \quad YZ = ZY.$$

Certainly there is a map ρ_1 from R onto $\text{Dhom}(G_1, [24])$:

$$\begin{aligned} \rho_1 : R &\longrightarrow \text{Dhom}(G_1, [24]) \\ X &\mapsto [12][24]^{-1} \\ Y &\mapsto [14][24]^{-1} \\ Z &\mapsto [23][24]^{-1}. \end{aligned}$$

As in the original case, we use Gelfand-Kirillov dimension to show that ρ_1 is an isomorphism.

Suppose that $\ker(\rho_1) \neq 0$ and note that $\text{GKdim}(R) = 3$. Then

$$\text{GKdim}(\text{Dhom}(G_1, [24])) = \text{GKdim}(R/\ker(\rho_1)) \leq 2.$$

The final inequality here is due to R being a domain. Also,

$$\begin{aligned} \text{GKdim}(\text{Dhom}(G_1, [24])) &= \text{GKdim}(T) - 1 \\ &= \text{GKdim}(G_1) - 1 \\ &= \text{GKdim}(G) - 2 \\ &= 3, \end{aligned}$$

and we have a contradiction. Therefore

$$\text{Dhom}(G_1, [24]) \cong R.$$

Note that since $\text{Dhom}(G_1, [24])$ is a domain (since R is a domain), Corollary 3.1.3 implies that G_1 is a domain.

Let $\sigma_{[24]}$ be the automorphism on $\text{Dhom}(G_1, [24])$ obtained by conjugating by $[24]$ and recall that we have a homeomorphism

$$\Gamma_{[24]} : \{P \in \text{GrSpec}(G_1) \mid [24] \notin P\} \longrightarrow \sigma_{[24]}\text{-Spec}(\text{Dhom}(G_1, [24])),$$

where $\Gamma_{[24]}(P) = PT \cap \text{Dhom}(G_1, [24])$. Thus, to find those graded primes in G_1 not containing $[24]$ (and therefore those graded primes in G containing $[34]$ but not $[24]$), we must identify the $\sigma_{[24]}$ -prime spectrum of R . In finding $\sigma_{[24]}\text{-Spec}(R)$ we make repeated use of the following result.

Proposition 4.2.5. [20] Proposition 14.2(ii).

Let $S = R[x; \sigma]$ where R is a noetherian ring and σ is an automorphism of R . The prime ideals of S that contain x are precisely those of the form $I + xS$ where $I \in \text{Spec}(R)$. \square

We use the commutation relations for G modulo [34] to calculate what $\sigma_{[24]}$ does to the generators of R :

$$\begin{aligned}\sigma_{[24]} : R &\longrightarrow R \\ X &\mapsto q^{-1}X \\ Y &\mapsto q^{-1}Y \\ Z &\mapsto q^{-1}Z.\end{aligned}$$

Note that we can write R as an Ore extension of the quantum plane, $k_q[X, Y]$; that is,

$$\begin{aligned}R = k_q[X, Y][Z, \tau_1]; \quad \tau_1 : X &\mapsto q^{-1}X \\ &Y \mapsto Y.\end{aligned}$$

Now suppose Q is a prime ideal of R and that $Z \in P$. Then by Proposition 4.2.5 P is an ideal of the form

$$Q = I + ZR$$

where $I \in \text{Spec}(k_q[X, Y])$. The prime spectrum of the quantum plane is well known:

$$\text{Spec}(k_q[X, Y]) = \{\langle 0 \rangle, \langle X \rangle, \langle Y \rangle, \langle X - \alpha, Y \rangle, \langle X, Y - \beta \rangle \mid \alpha, \beta \in k\}.$$

Thus

$$Q \in \{\langle Z \rangle, \langle X, Z \rangle, \langle Y, Z \rangle, \langle X - \alpha, Y, Z \rangle, \langle X, Y - \beta, Z \rangle \mid \alpha, \beta \in k\}$$

and it is clear that

$$\{(Q : \sigma_{[24]}) \mid Z \in Q \in \text{Spec}(R)\} = \{\langle Z \rangle, \langle X, Z \rangle, \langle Y, Z \rangle, \langle X, Y, Z \rangle\}.$$

Now suppose that $Y \in Q$ and note that we can rewrite R as an Ore extension of the quantum plane $k_q[X, Z]$:

$$\begin{aligned}R = k_q[X, Z][Y, \tau_2], \quad \tau_2 : X &\mapsto q^{-1}X \\ &Z \mapsto Z.\end{aligned}$$

Then, as above,

$$Q \in \{\langle Y \rangle, \langle X, Y \rangle, \langle Y, Z \rangle, \langle X - \alpha, Y, Z \rangle, \langle X, Y, Z - \beta \rangle \mid \alpha, \beta \in k\}$$

and hence

$$\{(Q : \sigma_{[24]}) \mid Y \in Q \in \text{Spec}(R)\} = \{\langle Y \rangle, \langle X, Y \rangle, \langle Y, Z \rangle, \langle X, Y, Z \rangle\}.$$

Now suppose that $X \in Q$ and notice that once again we can rewrite R , this time as an Ore extension of the plane $k[Y, Z]$; that is,

$$R = k[Y, Z][X; \tau_3], \quad \tau_3 : Y \mapsto qY \\ Z \mapsto qZ.$$

Then by Proposition 4.2.5 we have that

$$Q = I + XR$$

where I is a prime ideal of the plane $k[Y, Z]$. Now,

$$\text{Spec}(k[Y, Z]) = \{\langle 0 \rangle, \langle Y - \alpha, Z - \beta \rangle, \langle f \rangle \mid \alpha, \beta \in k\}$$

where $f \in k[Y, Z]$ is irreducible. Thus

$$\text{Spec}(k[Y, Z]) = \{\langle 0 \rangle, \langle Y - \alpha, Z - \beta \rangle, \langle f \rangle \cap k[Y, Z] \mid \alpha, \beta \in k\}$$

where $f \in k[Y^{\pm 1}, Z^{\pm 1}]$ is irreducible and

$$Q \in \{\langle X \rangle, \langle X, Y - \alpha, Z - \beta \rangle, \langle X, f \rangle \cap R \mid \alpha, \beta \in k\}$$

where $f \in k[Y^{\pm 1}, Z^{\pm 1}]$ is irreducible.

We must find $(Q : \sigma_{[24]})$ for each of the primes listed above. The only case here which causes any concern is when $Q = \langle X, f \rangle \cap R$. However, notice that the techniques employed in Case 1 can also be applied here to obtain,

$$(\langle X, f \rangle \cap R : \sigma_{[24]}) = \{\langle X, Y - \alpha Z \rangle \mid 0 \neq \alpha \in k\}.$$

Thus

$$\{(Q : \sigma_{[24]}) \mid X \in Q \in \text{Spec}(R)\} \\ = \{\langle X \rangle, \langle X, Y \rangle, \langle X, Z \rangle, \langle X, Y, Z \rangle, \langle X, Y - \alpha Z \rangle \mid 0 \neq \alpha \in k\}.$$

Finally suppose that $X, Y, Z \notin Q$. Then Q is a prime ideal of R not containing Y or Z . Thus by [20] 14.7(ii),

$$Q = IR$$

where I is a prime ideal of $k[Y, Z]$ not containing Y or Z . Therefore

$$I \in \{\langle 0 \rangle, \langle Z - \alpha, Y - \beta \rangle, \langle f \rangle \mid 0 \neq \alpha, \beta \in k\}$$

where $f \in k[Y, Z]$ is irreducible. Therefore

$$I \in \{\langle 0 \rangle, \langle Z - \alpha, Y - \beta \rangle, \langle f \rangle \cap k[Y, Z] \mid 0 \neq \alpha, \beta \in k\}$$

where $f \in k[Y^{\pm 1}, Z^{\pm 1}]$ is irreducible. Therefore

$$\{(Q : \sigma_{[24]}) \mid X, Y, Z \notin Q \in \text{Spec}(R)\} = \{\langle 0 \rangle, \langle Y - \alpha Z \rangle \mid 0 \neq \alpha \in k\}.$$

Combining the 4 cases ($Z \in P, Y \in P, X \in P, X, Y, Z \notin P$) we obtain

$$\begin{aligned} \sigma_{[24]}\text{-Spec}(R) &= \{(Q : \sigma_{[24]}) \mid P \in \text{Spec}(R)\} \\ &= \{\langle 0 \rangle, \langle X \rangle, \langle Y \rangle, \langle Z \rangle, \langle X, Y \rangle, \langle X, Z \rangle, \langle Y, Z \rangle, \\ &\quad \langle Y - \alpha Z \rangle, \langle X, Y - \alpha Z \rangle, \langle X, Y, Z \rangle \mid 0 \neq \alpha \in k\}. \end{aligned}$$

Then using ρ_1 we can write down the $\sigma_{[24]}$ -prime spectrum of $\text{Dhom}(G_1, [24])$. Let $\widetilde{[ij]} := [ij][24]^{-1}$. Then the poset of $\sigma_{[24]}$ -prime ideals in $\text{Dhom}(G_1, [24])$ can be seen in Figure 4.5 (where the dotted line indicates inclusion if and only if $\alpha = \beta$). Once again, we can use a preferred product argument to obtain $\Gamma_{[24]}^{-1}(I)$

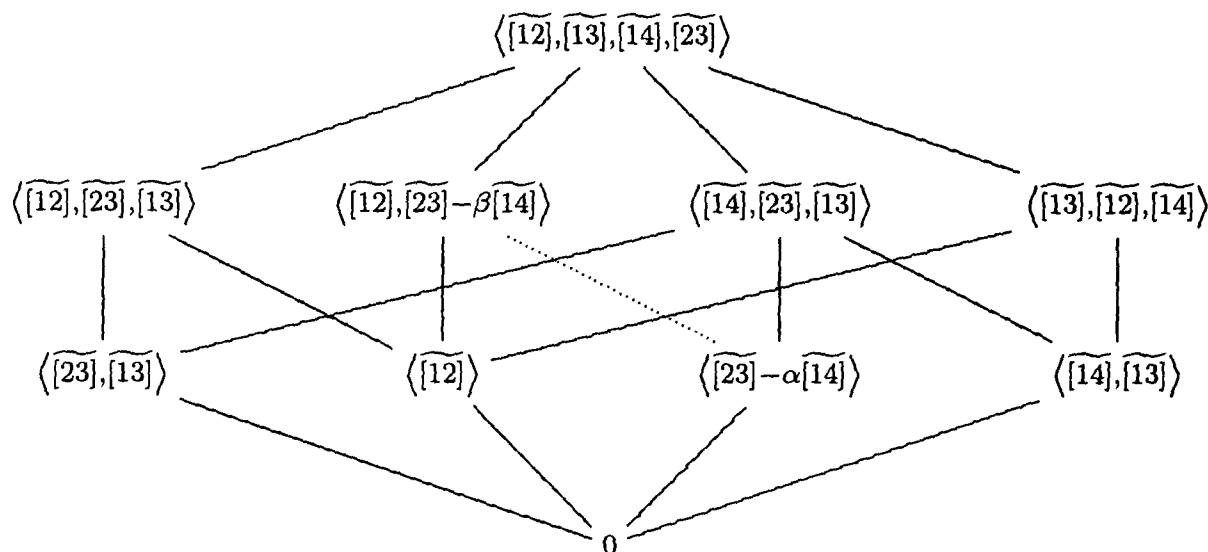


Figure 4.5: Poset of $\sigma_{[24]}$ -prime ideals in $\text{Dhom}(G_1, [24])$ ($\alpha, \beta \in k^\times$)

for all but two of the $\sigma_{[24]}$ -primes appearing in Figure 4.5. The resulting ideals are displayed in Figure 4.6 (where the dotted line indicates inclusion if and only if $\alpha = \beta$.)

Thus,

$$\{P \in \text{GrSpec}(G) \mid [34] \in P, [24] \notin P\} = \{([34], I) \mid I \in \{\text{GrSpec}(G_1) \mid [24] \notin I\}\}$$

and to see the poset of graded primes in G containing $[34]$ but not $[24]$, insert $[34]$ into the generating set of each in the primes in Figure 4.6.

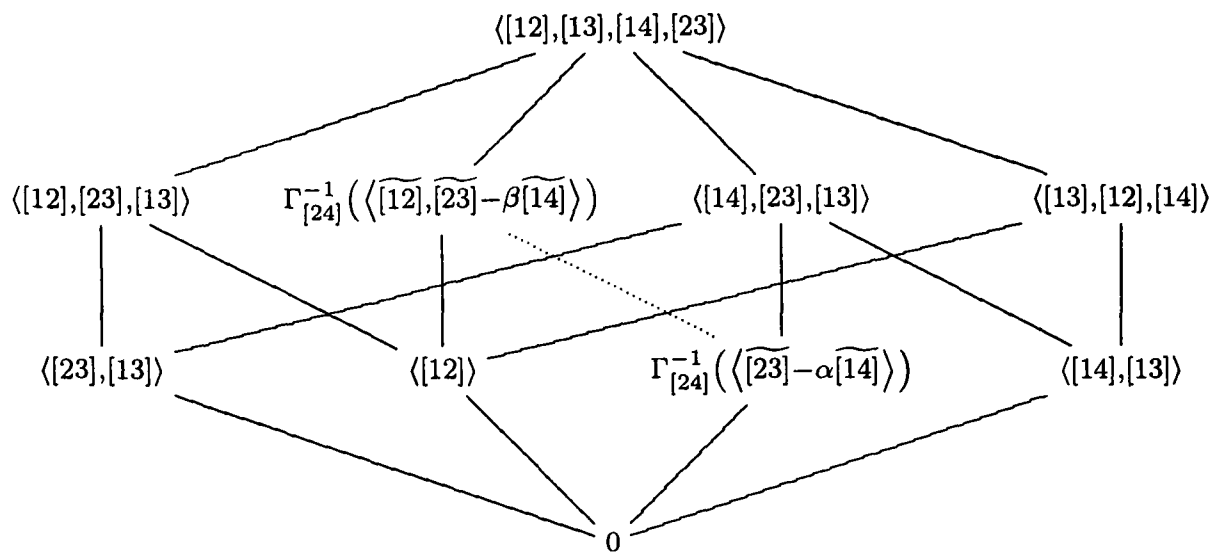


Figure 4.6: Poset of graded primes in G_1 not containing $[24]$ ($\alpha, \beta \in k^\times$)

Case 3: $[34], [24] \in P$, $[23] \notin P$

Consider the Quantum Plücker relation in G :

$$[12][34] - q[13][24] + q^2[14][23] = 0.$$

Then since $[34], [24] \in P$ we have that $[14][23] \in P$ and since $[23]$ is normal in G (this is easily seen from the relations given in Example 2.2.2), we have

$$[14]G[23]G \subseteq P.$$

Now, P is a graded prime ideal and $[23] \notin P$, therefore $[14]G \subseteq P$ and so $[14] \in P$. Thus, instead of factoring G by the ideal generated by $[34]$ and $[24]$, we should factor by the ideal generated by $[34], [24]$ and $[14]$. Let

$$G_2 := \frac{G}{\langle [34], [24], [14] \rangle}.$$

Then in G_2 we have the commutation relations

$$[12][13] = q[13][12], \quad [12][23] = q[23][12] \quad \text{and} \quad [13][23] = q[23][13].$$

Certainly $[23]$ is normal in G_2 . By a similar argument involving preferred products to that in Case 2 (where it is shown that $[24]$ is regular in G_1), we can show that $[23]$ is regular in G_2 . Therefore we may dehomogenise G_2 at $[23]$. Let

$$U := (G_2)_{[23]}.$$

Then U has an induced \mathbb{Z} -grading and $\text{Dhom}(G_2, [23]) = U_0$ is generated by the elements

$$[12][23]^{-1} \quad \text{and} \quad [13][23]^{-1},$$

which q -commute:

$$[12][23]^{-1}[13][23]^{-1} = q[13][23]^{-1}[12][23]^{-1}.$$

Thus there is a map ρ_2 from the quantum plane $k_q[x, y]$ onto $\text{Dhom}(G_2, [23])$:

$$\begin{aligned} \rho_2 : k_q[x, y] &\longrightarrow \text{Dhom}(G_2, [23]) \\ x &\mapsto [12][23]^{-1} \\ y &\mapsto [13][23]^{-1}. \end{aligned}$$

As in the previous cases we aim to show that ρ_2 is an isomorphism by using a Gelfand-Kirillov dimension argument. Clearly

$$\text{GKdim}(k_q[x, y]) = 2.$$

Now consider the Gelfand Kirillov dimension of G_2 . Since $G_2 \cong G_1 / \langle [24], [14] \rangle$ and $[24]$ is a normal non-zero divisor in G_1 , we have that

$$\begin{aligned} \text{GKdim}(G_2) = \text{GKdim}(G_1 / \langle [24], [14] \rangle) &\leq \text{GKdim}(G_1) - 1 \\ &= \text{GKdim}(G) - 2 \\ &= 3. \end{aligned}$$

Suppose that $\text{GKdim}(G_2) \leq 2$. Inspection of the commutation relations for G_2 and our usual argument for proving regularity of elements, obtains for us a regular normalising sequence of elements $\{[23], [13], [12]\}$ in G_2 . Therefore

$$\begin{aligned} \text{GKdim}(k) = \text{GKdim}\left(\frac{G_2}{\langle [23], [13], [12] \rangle}\right) &= \text{GKdim}(G_2) - 3 \\ &< 0. \end{aligned}$$

Therefore $\text{GKdim}(G_2) = 3$ and $\text{GKdim}(\text{Dhom}(G_2, [23])) = 2$. Now if $\ker \rho_2 \neq 0$, then

$$\begin{aligned} \text{GKdim}(\text{Dhom}(G_2, [23])) &= \text{GKdim}(k_q[x, y] / \ker \rho_2) \\ &\leq \text{GKdim}(k_q[x, y]) - 1 \\ &= 1, \end{aligned}$$

which is a contradiction. Therefore $\ker \rho_2 = 0$ and ρ_2 is an isomorphism. Thus $\text{Dhom}(G_2, [23])$ is a domain and so G_2 is a domain by Corollary 3.1.3.

Let $\sigma_{[23]} : \text{Dhom}(G_2, [23]) \rightarrow \text{Dhom}(G_2, [23])$ be the automorphism obtained by conjugating by $[23]$ and recall the homeomorphism

$$\Gamma_{[23]} : \{P \in \text{GrSpec}(G_2) \mid [23] \notin P\} \rightarrow \sigma_{[23]}\text{-Spec}(\text{Dhom}(G_2, [23]))$$

such that $\Gamma_{[23]}(P) = PU \cap \text{Dhom}(G_2, [23])$. In order to identify those primes in G_2 not containing $[23]$, we should find the $\sigma_{[23]}$ -prime spectrum of $\text{Dhom}(G_2, [23])$. Equivalently, we identify the $\sigma_{[23]}$ -prime spectrum of the quantum plane. We have already seen (in Case 2) that the prime spectrum of the quantum plane has been completely described:

$$\text{Spec}(k_q[x, y]) = \{\langle 0 \rangle, \langle x \rangle, \langle y \rangle, \langle x - \alpha, y \rangle, \langle x, y - \beta \rangle \mid \alpha, \beta \in k\}.$$

By Proposition 4.2.4

$$\sigma_{[23]}-\text{Spec}(k_q[x, y]) = \{(P : \sigma_{[23]}) \mid P \in \text{Spec}(k_q[x, y])\}.$$

Now, from the commutation relations for G_2 ,

$$\begin{aligned} \sigma_{[23]} : k_q[x, y] &\longrightarrow k_q[x, y] \\ x &\mapsto q^{-1}x \\ y &\mapsto q^{-1}y. \end{aligned}$$

Therefore

$$\sigma_{[23]}-\text{Spec}(k_q[x, y]) = \{\langle 0 \rangle, \langle x \rangle, \langle y \rangle, \langle x, y \rangle\}.$$

Thus

$$\sigma_{[23]}-\text{Spec}(\text{Dhom}(G_2, [23])) =$$

$$\{\langle 0 \rangle, \langle [12][23]^{-1} \rangle, \langle [13][23]^{-1} \rangle, \langle [12][23]^{-1}, [13][23]^{-1} \rangle\}.$$

As in Cases 1 and 2, we can use a preferred product argument to obtain those graded ideals in G_2 not containing $[23]$:

$$\{P \in \text{GrSpec}(G_2) \mid [23] \notin P\} = \{\langle 0 \rangle, \langle [12] \rangle, \langle [13] \rangle, \langle [12], [13] \rangle\}.$$

Therefore

$$\{P \in \text{GrSpec}(G) \mid [34], [24], [14] \in P, [23] \notin P\} =$$

$$\begin{aligned} &\{\langle [14], [24], [34] \rangle, \langle [14], [24], [34], [12] \rangle, \\ &\quad \langle [14], [24], [34], [13] \rangle, \langle [14], [24], [34], [12], [13] \rangle\}. \end{aligned}$$

Case 4: $[34], [24], [23] \in P, [14] \notin P$

As we proceed, the reader should notice the similarity between this case and Case 3.

We factor G by the ideal generated by $[34], [24], [23]$:

$$G_3 := \frac{G}{\langle [34], [24], [23] \rangle}.$$

In G_3 we have the following commutation relations:

$$[12][13] = q[13][12], \quad [12][14] = q[14][12], \quad [13][14] = q[13][12].$$

Our usual argument establishes that $[14]$ is a regular normal element in G_3 and we may dehomogenise G_3 at $[14]$. Let

$$V := (G_3)_{[14]}.$$

Then V has an induced \mathbb{Z} -grading and $\text{Dhom}(G_3, [14]) = V_0$ is generated by

$$[12][14]^{-1} \quad \text{and} \quad [13][14]^{-1},$$

which are subject to the relation

$$[12][14]^{-1}[13][14]^{-1} = q[13][14]^{-1}[12][14]^{-1}.$$

Thus, there is a map ρ_3 from the quantum plane $k_q[x, y]$ onto $\text{Dhom}(G_3, [14])$:

$$\begin{aligned} \rho_3 : k_q[x, y] &\longrightarrow \text{Dhom}(G_3, [14]) \\ x &\mapsto [12][14]^{-1} \\ y &\mapsto [13][14]^{-1}. \end{aligned}$$

By a similar argument to that in Case 3, ρ_3 is in fact an isomorphism:

$$\text{Dhom}(G_3, [14]) \cong k_q[x, y].$$

Thus $\text{Dhom}(G_3, [14])$ is a domain and so G_3 is a domain by Corollary 3.1.3.

Let $\sigma_{[14]}$ be the automorphism obtained by conjugating by $[14]$. Then

$$\begin{aligned} \sigma_{[14]} : k_q[x, y] &\longrightarrow k_q[x, y] \\ x &\mapsto q^{-1}x \\ y &\mapsto q^{-1}y. \end{aligned}$$

Therefore, from Case 3,

$$\sigma_{[14]}-\text{Spec}(k_q[x, y]) = \{\langle 0 \rangle, \langle x \rangle, \langle y \rangle, \langle x, y \rangle\}.$$

Thus

$$\sigma_{[14]}-\text{Spec}(\text{Dhom}(G_3, [14])) =$$

$$\{\langle 0 \rangle, \langle [12][14]^{-1} \rangle, \langle [13][14]^{-1} \rangle, \langle [12][14]^{-1}, [13][14]^{-1} \rangle\}.$$

Then using the map

$$\Gamma_{[14]}^{-1} : \sigma_{[14]}\text{-Spec}(\text{Dhom}(G_3, [14])) \longrightarrow \{P \in \text{GrSpec}(G_3) \mid [14] \notin P\},$$

given by $\Gamma_{[14]}^{-1}(P) = PV \cap G_4$, and a preferred product argument similar to that used in Case 1, we obtain

$$\{P \in G_3 \mid [14] \notin P\} = \{\langle 0 \rangle, \langle [12] \rangle, \langle [13] \rangle, \langle [12], [13] \rangle\}.$$

Thus

$$\begin{aligned} \{P \in G \mid [34], [24], [23] \in P, [14] \notin P\} = \\ \{\langle [34], [24], [23] \rangle, \langle [34], [24], [23], [12] \rangle, \\ \langle [34], [24], [23], [13] \rangle, \langle [34], [24], [23], [13], [12] \rangle\}. \end{aligned}$$

Case 5: $[34], [24], [23], [14] \in P, [13] \notin P$

We factor G by the ideal generated by $[34], [24], [23]$ and $[14]$. So

$$G_4 := \frac{G}{\langle [34], [24], [23], [14] \rangle}.$$

Then certainly $[13]$ is a regular normal element of G_4 and thus we may dehomogenise G_4 at $[13]$. Let

$$W := (G_4)_{[13]}.$$

Then W is \mathbb{Z} -graded and $\text{Dhom}(G_4, [13]) = W_0$ is generated by $[12][13]^{-1}$. There is a map from $k[x]$ onto $\text{Dhom}(G_4, [13])$:

$$\begin{aligned} \rho_4 : k[x] &\longrightarrow \text{Dhom}(G_4, [13]) \\ x &\mapsto [12][13]^{-1}. \end{aligned}$$

Now, $\text{GKdim}(k[x]) = 1$ and since $G_4 \cong G_3 / \langle [14] \rangle$, we have that $\text{GKdim}(G_4) = 2$ and therefore $\text{GKdim}(\text{Dhom}(G_4, [13])) = 1$ and ρ_4 is an isomorphism.

Let $\sigma_{[13]} : \text{Dhom}(G_4, [13]) \rightarrow \text{Dhom}(G_4, [13])$ be the automorphism obtained by conjugating by $[13]$. Then

$$\begin{aligned} \sigma_{[13]} : k[x] &\longrightarrow k[x] \\ x &\mapsto q^{-1}x. \end{aligned}$$

We now identify the $\sigma_{[13]}$ -prime spectrum of $k[x]$ and therefore the $\sigma_{[13]}$ -prime spectrum of $\text{Dhom}(G_4, [13])$. We then use the map

$$\Gamma_{[13]}^{-1} : \sigma_{[13]}-\text{Spec}(\text{Dhom}(G_4, [13])) \longrightarrow \{P \in \text{GrSpec}(G_4) \mid [13] \notin P\}$$

to obtain the graded primes in G_4 not containing $[13]$. The prime spectrum of $k[x]$ is well known:

$$\text{Spec}(k[x]) = \{\langle 0 \rangle, \langle x - \alpha \rangle \mid \alpha \in k\}.$$

So by Proposition 4.2.4,

$$\sigma_{[13]}-\text{Spec}(k[x]) = \{\langle 0 \rangle, \langle x \rangle\}$$

and

$$\sigma_{[13]}-\text{Spec}(\text{Dhom}(G_4, [13])) = \{\langle 0 \rangle, \langle [12][13]^{-1} \rangle\}.$$

By using a preferred product argument similar to that in Case 1 we obtain

$$\{P \in \text{GrSpec}(G_4) \mid [13] \notin P\} = \{\langle 0 \rangle, \langle [12] \rangle\}.$$

Therefore

$$\begin{aligned} \{P \in \text{GrSpec}(G) \mid [34], [24], [23], [14] \in P, [13] \notin P\} = \\ \{\langle [34], [24], [23], [14] \rangle, \langle [34], [24], [23], [14], [12] \rangle\}. \end{aligned}$$

Case 6: $[34], [24], [23], [14], [13] \in P, [12] \notin P$

First we factor G by those minors which sit inside the ideal P . Let

$$G_5 := \frac{G}{\langle [34], [24], [23], [14], [13] \rangle}.$$

Then $[12]$ is a regular normal element in G_5 . Let Y be the localisation of G_5 at $[12]$. Then Y is \mathbb{Z} -graded and $\text{Dhom}(G_5, [12]) = Y_0$ is generated by $[12][12]^{-1} = 1$. Therefore $\text{Dhom}(G_5, [12])$ is a copy of the field k . From this point it is easy to see that

$$\begin{aligned} \{P \in \text{GrSpec}(G) \mid [34], [24], [23], [14], [13] \in P, [12] \notin P\} = \\ \{\langle [34], [24], [23], [14], [13] \rangle\}. \end{aligned}$$

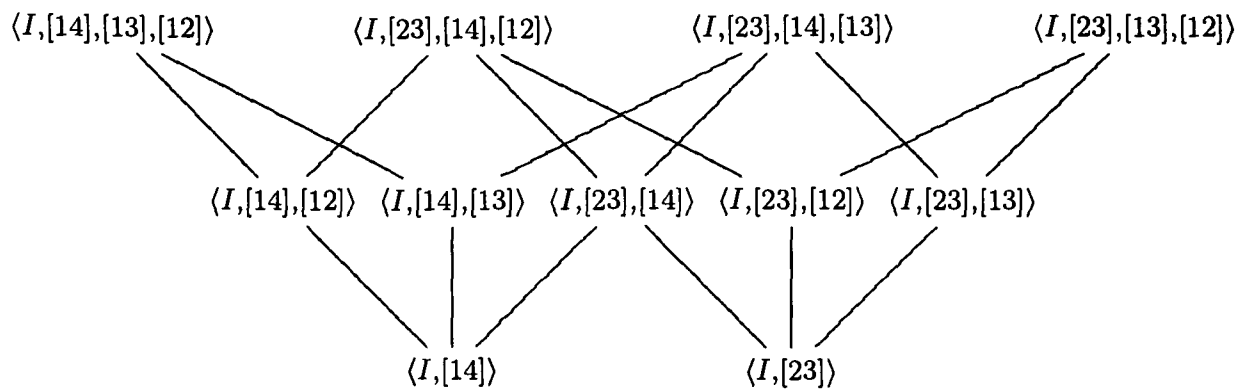


Figure 4.7: Poset of graded primes of $G_q(2, 4)$ containing $[24]$ and $[34]$

We can get some feeling for how the graded primes obtained in Cases 3 to 6 fit together by combining them in a diagram. Figure 4.7 shows the partially ordered set of graded primes in $G_q(2, 4)$ containing $[34]$ and $[24]$, where $I = \{[34], [24]\}$. Certainly, a diagram showing the poset of all graded prime ideals of $G_q(2, 4)$ would be desirable, but rather too complicated for us to display.

Having worked through the 6 cases, we are now able to explicitly write down the graded prime spectrum of $G_q(2, 4)$ as a disjoint partition of ‘cells’:

$$\text{GrSpec}(G_q(2, 4)) = Q^4 \sqcup Q^3 \sqcup Q^2 \sqcup Q^2 \sqcup Q^1 \sqcup Q^0$$

where

$Q^4 \approx \sigma_{[34]}\text{-Spec}(\mathcal{O}_q(M_2))$ - $\sigma_{[34]}$ -prime spectrum of 2×2 quantum matrices,

$Q^3 \approx \sigma_{[24]} \text{-Spec}(R)$ - $\sigma_{[24]}$ -prime spectrum of a ‘quantum 3-space’,

$Q^2 \approx \sigma_{[23]} \text{-Spec}(k_q[x, y])$ - $\sigma_{[23]}/\sigma_{[14]}$ -prime spectrum of the quantum plane,

$Q^1 \approx \sigma_{[13]} \text{-Spec}(k[x])$ - $\sigma_{[13]}$ -prime spectrum of a ‘quantum 1-space’,

$Q^0 \approx \sigma_{[12]} \text{-Spec}(k)$ - $\sigma_{[12]}$ -prime spectrum of a ‘quantum 0-space’.

Thus we have produced a ‘cell decomposition’ of $G_q(2, 4)$ (or more precisely of $\text{GrSpec}(G_q(2, 4))$) which mimics exactly the cell decomposition of $\mathcal{O}(G(2, 4))$ in the classical case. In the process of finding this cell decomposition, we have also completely described the graded prime spectrum of the 2×4 quantum Grassmannian.

Recall the following definition.

Definition 4.2.6. A ring R has **normal separation** if for all primes P, Q such that $Q \subsetneq P$, there exists $u \in P \setminus Q$ such that u is normal modulo Q .

Of course, we have a similar definition in the graded set-up. Let R be a graded ring. Then R has **graded normal separation** if for all graded primes P, Q such

that $Q \subsetneq P$, there exists $u \in P \setminus Q$ such that u is homogeneous and normal modulo Q . Goodearl [12] has shown that when we have a graded noetherian ring which has graded normal separation, then it also has normal separation ([12] Corollary 4.6).

Proposition 4.2.7. *Let R be a \mathbb{Z}^n -graded noetherian ring. If R has graded normal separation, then R has normal separation.* \square

We will show that $G_q(2, 4)$ has graded normal separation. Consider $P, Q \in \text{GrSpec}(G_q(2, 4))$ such that $Q \subsetneq P$. Let $\mathcal{L} = \{[12], [13], [14], [23], [24], [34]\}$ and recall that we have a total lexicographic ordering on \mathcal{L} :

$$[12] <_{\text{lex}} [13] <_{\text{lex}} [14] <_{\text{lex}} [23] <_{\text{lex}} [24] <_{\text{lex}} [34].$$

There are two cases to consider.

1. $P \cap \mathcal{L} \neq Q \cap \mathcal{L}$

Then there exists $[ij] \in P \cap \mathcal{L}$ such that $[ij] \notin Q \cap \mathcal{L}$. Let $[ij]$ be the least such minor with respect to the lexicographical ordering. Then all minors beneath $[ij]$ in the lexicographical ordering are contained in Q . By Lemma 2.2.5 any minor is normal modulo the ideal generated by the set $\{[rs] \mid [rs] <_{\text{lex}} [ij]\}$. Therefore $[ij]$ is normal modulo Q .

2. $P \cap \mathcal{L} = Q \cap \mathcal{L}$

In this case P and Q must belong to the same ‘cell’ in the decomposition of $G_q(2, 4)$. Since $P \cap \mathcal{L} = Q \cap \mathcal{L}$ and $P \neq Q$, either P or Q must contain a generator which is not a single minor. Inspection of the diagrams, Figures 4.4, 4.6 (with $[34]$ inserted into the generating set of each of the ideals appearing) and 4.7, displaying the graded primes of $G_q(2, 4)$, reveals that there are number of ways in which this can happen:

- (i) $P = \Gamma_{[34]}^{-1} \left(\langle \widetilde{[23]} - \alpha \widetilde{[14]} \rangle \right)$, $Q = \langle 0 \rangle$,
- (ii) $P = \Gamma_{[34]}^{-1} \left(\langle \widetilde{[12]}, \widetilde{[23]} - \alpha \widetilde{[14]} \rangle \right)$, $Q = \langle [12] \rangle$,
- (iii) $P = \Gamma_{[34]}^{-1} \left(\langle \widetilde{[23]}, \widetilde{[14]}, \widetilde{[13]} - \gamma \widetilde{[24]} \rangle \right)$, $Q = \Gamma_{[34]}^{-1} \left(\langle \widetilde{[23]}, \widetilde{[14]} \rangle \right)$,
- (iv) $P = \Gamma_{[24]}^{-1} \left(\langle \widetilde{[23]} - \alpha \widetilde{[14]} \rangle \right) + \langle [34] \rangle$, $Q = \langle [34] \rangle$,
- (v) $P = \Gamma_{[24]}^{-1} \left(\langle \widetilde{[12]}, \widetilde{[23]} - \alpha \widetilde{[14]} \rangle \right) + \langle [34] \rangle$, $Q = \langle [34], [12] \rangle$.

In each of these cases we must find a homogeneous element $u \in P \setminus Q$ such that

u is normal modulo Q . Consider $[23] - \alpha[14]$ in $G_q(2, 4)$. Then

$$\begin{aligned} [12]([23] - \alpha[14]) &= q([23] - \alpha[14])[12], \\ [13]([23] - \alpha[14]) &= q([23] - \alpha[14])[13], \\ [14]([23] - \alpha[14]) &= ([23] - \alpha[14])[14], \\ [23]([23] - \alpha[14]) &= ([23] - \alpha[14])[23], \\ [24]([23] - \alpha[14]) &= q^{-1}([23] - \alpha[14])[24], \\ [34]([23] - \alpha[14]) &= q^{-1}([23] - \alpha[14])[34]. \end{aligned}$$

In each of the cases (i), (ii), (iv) and (v), $[23] - \alpha[14] \in P \setminus Q$ and thus in these cases we take $u = [23] - \alpha[14]$. Now consider case (iii). Then modulo Q we have the commutation relations

$$\begin{aligned} [12]([13] - \gamma[24]) &= q([13] - \gamma[24])[12], \\ [13]([13] - \gamma[24]) &= ([13] - \gamma[24])[13], \\ [24]([13] - \gamma[24]) &= ([13] - \gamma[24])[24], \\ [34]([13] - \gamma[24]) &= q^{-1}([13] - \gamma[24])[34]. \end{aligned}$$

So $[13] - \gamma[24]$ is normal modulo Q and in this case we take $u = [13] - \gamma[24]$.

Therefore $G_q(2, 4)$ has graded normal separation and we obtain the following result.

Proposition 4.2.8. *The 2×4 quantum Grassmannian has normal separation.* \square

Let R be a ring and P, P' be prime ideals of R such that $P \supsetneq P'$. Recall that a chain of primes between P and P'

$$P = P_0 \supsetneq P_1 \supsetneq \dots \supsetneq P_n = P'$$

is **saturated** if no additional terms can be inserted. The ring R is said to be **catenary** if given any two primes $P \supsetneq P'$, any two saturated chains of primes between P and P' have the same length. The following result appears in [17] Theorem 1.6.

Theorem 4.2.9. *Let R be an affine noetherian Auslander Gorenstein and Cohen Macaulay k -algebra with finite Gelfand-Kirillov dimension. If R has normal separation, then R is catenary.* \square

We have shown that the 2×4 quantum Grassmannian satisfies the hypotheses of Theorem 4.2.9 and thus we have the following result.

Corollary 4.2.10. *The 2×4 quantum Grassmannian, $G_q(2, 4)$, is catenary. \square*

We also conjecture that the general quantum Grassmannian $G_q(m, n)$ has normal separation. In view of our earlier conjecture that $G_q(m, n)$ is Auslander Gorenstein and Cohen Macaulay, we have the following conjecture.

Conjecture 4.2.11. *The $m \times n$ quantum Grassmannian, $G_q(m, n)$, is catenary. \square*

Chapter 5

Auslander Regular Rings

Let R be a ring and let $x \in R$ be a regular central element. The relationship between the homological properties of the ring and those of the factor ring R/Rx , and the localisation of R at x , has been studied in [29] and [30]. In particular the following result ([30] Theorem 3.3.6) is obtained.

Theorem 5.0.1. *Let R be a noetherian ring and let $x \in R$ be a regular central element. If R/Rx and R_x are Auslander Regular, then R is Auslander Regular.*

□

In a noncommutative ring it is not always possible to find a regular central element. Of course the ‘next best thing’ is to have a regular normal element; recall that $x \in R$ is normal if $xR = Rx$. The aim of this chapter is to prove the equivalent result to Theorem 5.0.1 for $x \in R$ a regular normal element.

In the case that R is a connected \mathbb{N} -graded noetherian k -algebra and $x \in R$ is homogeneous, we have from Theorem 1.5.11 that R/Rx is Auslander Gorenstein if and only if R is. Thus, in this case, it only remains to show that if R/Rx and R_x are Auslander Regular, then R has finite global dimension. In fact, we will show in Section 5.1 that for an arbitrary noetherian ring R with a regular normal element $x \in R$, we have that

$$\text{gldim}(R) \leq \max\{\text{gldim}(R/Rx) + 1, \text{gldim}(R_x)\},$$

with equality if the global dimension of the factor ring is finite.

Let R be an arbitrary noetherian ring and let $x \in R$ be a regular normal element. In Section 5.2 the aim is to show that if R/Rx and R_x are Auslander Regular, then R is Auslander Regular. Clearly the result above says that R will have finite global dimension, thus we need to show that R satisfies the Auslander Condition.

We achieve this by considering results of Li HuiShi and Van Oystaeyen from [31] Chapter III for the case that $x \in R$ is central, and deriving equivalent results for the case that $x \in R$ is only normal.

5.1 The Global Dimensions of Related Rings

Recall the following definitions from 1.5.10.

The **grade** of a finitely generated (right or left) R -module M is defined by

$$j_R(M) = \inf\{i \mid \text{Ext}_R^i(M, R) \neq 0\} \in \mathbb{N} \cup \{\infty\}.$$

The Auslander condition: The ring R satisfies the Auslander condition if for each finitely generated R -module M , and for all $i \geq 0$ and every R -submodule N of $\text{Ext}_R^i(M, R)$, we have $j_R(N) \geq i$.

A ring of finite injective dimension which satisfies the Auslander condition is called **Auslander Gorenstein**.

A ring of finite global dimension which satisfies the Auslander condition is called **Auslander Regular**.

Let x be a regular normal element of a noetherian ring R . The aim in this section is to show that if R/Rx and R_x have finite global dimension, then R has finite global dimension. The following result appears in [29] Lemma 2.1.

Lemma 5.1.1. *Let x be a regular central element of a noetherian ring R . Then*

$$\text{gldim}(R) \leq \max\{\text{gldim}(R/Rx) + 1, \text{gldim}(R_x)\}$$

with equality if $\text{gldim}(R/Rx)$ is finite. □

A study of this Lemma indicates that its proof relies on two main steps; the first is an inductive step based on the following result ([29] Lemma 2.1).

Lemma 5.1.2. *Let x be a regular central element of a noetherian ring R , and let M be a finitely generated x -torsion free right R -module. If M/Mx is a projective right R/Rx -module, and $M_x = M \otimes_R R_x$ is a projective R_x -module, then M is a projective R -module.* □

The second step is actually in the proof of this Lemma and relies on the following well known result from [37] Theorem 7.16.

Theorem 5.1.3. *Let R be a ring, $x \in R$ be central and M be a right R -module. If $\mu : M \rightarrow M$ is multiplication by x , then $\mu^* : \text{Ext}_R^n(M, R) \rightarrow \text{Ext}_R^n(M, R)$ is also multiplication by x . \square*

When x is only normal, multiplication by x is not even a module map. However, we have already seen a way to sidestep this problem. Recall (Definition 1.3.11) that if $\sigma : R \rightarrow R$ is an automorphism and M is a right R -module, then M^σ is the right R -module with the same underlying abelian group as M , but with module multiplication given by

$$m *_\sigma r = m\sigma(r).$$

Thus in order to use Theorem 5.1.3 (or at least a version of it) we should ‘twist’ the module action and adjust the result accordingly. First we should familiarise ourselves with the effect that twisting the module action has on the module homomorphisms.

Lemma 5.1.4. *Let R be a ring, $\sigma : R \rightarrow R$ be an automorphism and M and N be right R -modules. Suppose we have a homomorphism*

$$\rho : M \rightarrow N.$$

Then we also have a homomorphism

$$\rho^\sigma : M^\sigma \rightarrow N^\sigma, \text{ defined by } \rho^\sigma(m^\sigma) := \rho(m) \text{ for all } m^\sigma \in M^\sigma,$$

such that $\ker(\rho^\sigma) = \ker(\rho)$, and $\text{im}(\rho^\sigma) = \text{im}(\rho)$ as abelian groups.

Moreover, if $\dots \rightarrow P_1 \xrightarrow{d_1} P_0 \xrightarrow{\epsilon} M \rightarrow 0$ is a projective resolution for M , then $\dots \rightarrow P_1^\sigma \xrightarrow{d_1^\sigma} P_0^\sigma \xrightarrow{\epsilon^\sigma} M^\sigma \rightarrow 0$ is a projective resolution for M^σ .

Note. As sets, M and M^σ are the same, and as set maps $\rho = \rho^\sigma$. We distinguish the two maps to keep track of which module action we are using.

Proof. Clearly ρ^σ is additive. Let $r \in R$ and $m^\sigma \in M^\sigma$. Then

$$\begin{aligned} \rho^\sigma(m^\sigma *_\sigma r) &= \rho(m\sigma(r)) \\ &= \rho(m)\sigma(r) \\ &= \rho^\sigma(m^\sigma) *_\sigma r. \end{aligned}$$

Thus $\rho^\sigma : M^\sigma \rightarrow N^\sigma$ is an R -module homomorphism and clearly $\ker(\rho^\sigma) = \ker(\rho)$ and $\text{im}(\rho^\sigma) = \text{im}(\rho)$ as abelian groups. Therefore

$$\dots \rightarrow P_1^\sigma \xrightarrow{d_1^\sigma} P_0^\sigma \xrightarrow{\epsilon^\sigma} M^\sigma \rightarrow 0$$

is an exact sequence. It remains to show that if P is a projective R -module, then P^σ is a projective R -module. Suppose we have the usual diagram

$$\begin{array}{ccc} & P^\sigma & \\ \exists \gamma? \swarrow & \downarrow \alpha & \\ B & \xrightarrow{\beta} & C \longrightarrow 0. \end{array}$$

To show that P^σ is projective we must show that γ exists and the diagram commutes. Applying σ^{-1} to the previous diagram we get

$$\begin{array}{ccc} & (P^\sigma)^{\sigma^{-1}} & \\ & \downarrow \alpha^{\sigma^{-1}} & \\ B^{\sigma^{-1}} & \xrightarrow{\beta^{\sigma^{-1}}} & C^{\sigma^{-1}} \longrightarrow 0, \end{array}$$

and since P is projective we have

$$\begin{array}{ccc} & P & \\ \gamma' \swarrow & \downarrow \alpha^{\sigma^{-1}} & \\ B^{\sigma^{-1}} & \xrightarrow{\beta^{\sigma^{-1}}} & C^{\sigma^{-1}} \longrightarrow 0, \end{array}$$

with $\beta^{\sigma^{-1}} \gamma' = \alpha^{\sigma^{-1}}$. Thus, applying σ , we get

$$\begin{array}{ccc} & P^\sigma & \\ \gamma = (\gamma')^\sigma \swarrow & \downarrow \alpha & \\ B & \xrightarrow{\beta} & C \longrightarrow 0, \end{array}$$

with $\beta \gamma = \alpha$, where $\gamma = (\gamma')^\sigma$. □

Lemma 5.1.5. *Let R be a ring and $\sigma : R \rightarrow R$ be an automorphism. Then*

$$\begin{aligned} \rho : R_R &\rightarrow R_R^\sigma \\ r &\mapsto \sigma(r) \end{aligned}$$

is a right R -module isomorphism.

Proof Clearly ρ is an additive map which is both surjective and injective. It remains to show that ρ is a module homomorphism. Let $r \in R$ and $s \in R_R$.

Then

$$\begin{aligned}
 \rho(sr) &= \sigma(sr) \\
 &= \sigma(s)\sigma(r) \\
 &= \rho(s)\sigma(r) \\
 &= \rho(s) *_{\sigma} r,
 \end{aligned}$$

as required. \square

Lemma 5.1.6. *Let R be a ring and $\sigma : R \rightarrow R$ be an automorphism. Then for any finitely generated right R -module M :*

$$\mathrm{Hom}_R(M^{\sigma}, R) \cong {}^{\sigma}[\mathrm{Hom}_R(M, R)]$$

via the R -module homomorphism

$$\theta : f \mapsto \{\theta(f) : m \mapsto \sigma(f(m))\}.$$

Moreover θ induces an isomorphism

$$\bar{\theta} : \mathrm{Ext}_R^n(M^{\sigma}, R) \rightarrow {}^{\sigma}[\mathrm{Ext}_R^n(M, R)]$$

for $n \geq 0$.

Remarks.

1. $\mathrm{Hom}_R(M, R)$ is a left R -module with module multiplication $r.f : m \mapsto rf(m)$, $f \in \mathrm{Hom}_R(M, R)$, $r \in R$, $m \in M$.
2. ${}^{\sigma}[\mathrm{Hom}_R(M, R)]$ is the left R -module with the same underlying abelian group as $\mathrm{Hom}_R(M, R)$, but with multiplication given by $r_{\sigma} * f := \sigma(r).f$.

Similar remarks hold for $\mathrm{Ext}_R^n(M, R)$ and ${}^{\sigma}[\mathrm{Ext}_R^n(M, R)]$.

Proof. The map θ is easily seen to be additive. We check that θ is an R -module map. Let $r \in R$, $m \in M$ and $f \in \mathrm{Hom}_R(M^{\sigma}, R)$. Then

$$\begin{aligned}
 \theta(r.f)(m) &= \sigma((r.f)(m)) \\
 &= \sigma(rf(m)) \\
 &= \sigma(r)\sigma(f(m)) \\
 &= [\sigma(r).\theta(f)](m) \\
 &= [r_{\sigma} * \theta(f)](m),
 \end{aligned}$$

as required.

Now suppose $f \in \ker \theta$; that is, $\theta(f)(m) = 0$ for all $m \in M$. Then $\sigma(f(m)) = 0$

and so $f(m) = 0$, since σ is an automorphism. Therefore θ is injective.

Now let $g \in {}^\sigma[\text{Hom}_R(M, R)]$, so $g : M \rightarrow R$. Then by Lemma 5.1.4, we have $g^\sigma : M^\sigma \rightarrow R^\sigma$ such that $g^\sigma(m^\sigma) = g(m)$, for all $m^\sigma \in M^\sigma$. Also, recall from Lemma 5.1.5 the module isomorphism

$$\rho^{-1} : R_R^\sigma \rightarrow R_R \text{ such that } \rho^{-1}(r^\sigma) = \sigma^{-1}(r), \text{ for all } r^\sigma \in R_R^\sigma.$$

Then $\rho^{-1}g^\sigma \in \text{Hom}_R(M^\sigma, R)$ and

$$\begin{aligned} \theta(\rho^{-1}g^\sigma)(m^\sigma) &= \sigma(\rho^{-1}(g(m))) \\ &= \sigma(\sigma^{-1}(g(m))) \\ &= g(m), \end{aligned}$$

for all $m^\sigma \in M^\sigma$. Therefore θ is an isomorphism.

Now let $\dots \rightarrow P_1 \xrightarrow{d_1} P_0 \xrightarrow{\epsilon} M \rightarrow 0$ be a projective resolution for M . Then $\dots \rightarrow P_1^\sigma \xrightarrow{d_1^\sigma} P_0^\sigma \xrightarrow{\epsilon^\sigma} M^\sigma \rightarrow 0$ is a projective resolution for M^σ and we have isomorphisms

$$\begin{aligned} \theta_n : \text{Hom}_R(P_n^\sigma, R) &\rightarrow {}^\sigma[\text{Hom}_R(P_n, R)] \\ f &\mapsto \{\theta_n(f) : p \mapsto \sigma(f(p))\}. \end{aligned}$$

Let $T := \text{Hom}_R(-, R)$ and recall that

$$\text{Ext}_R^n(M, R) = \frac{\ker(Td_{n+1})}{\text{im}(Td_n)}.$$

Define

$$\begin{aligned} \tilde{\theta} : \ker(Td_{n+1}^\sigma) &\rightarrow {}^\sigma[\text{Ext}_R^n(M, R)] \\ f &\mapsto \theta_n(f) + {}^\sigma[\text{im}(Td_n)]. \end{aligned}$$

Clearly, $\tilde{\theta}$ is a surjective R -module homomorphism and

$$\ker(\tilde{\theta}) = \{f \in \ker(Td_{n+1}^\sigma) \mid \theta_n(f) \in {}^\sigma[\text{im}(Td_n)]\}.$$

Claim: $\ker(\tilde{\theta}) = \text{im}(Td_n^\sigma)$

Suppose $f \in \ker(\tilde{\theta})$. Then $\theta_n(f) \in \text{im}(Td_n)$ and there exists $g \in \text{Hom}_R(P_{n-1}, R)$ such that $\theta_n(f) = Td_n(g)$; that is, $\theta_n(f) = gd_n$.

Now, by Lemma 5.1.4 there exists $g^\sigma \in \text{Hom}_R(P_{n-1}^\sigma, R^\sigma)$ such that $g^\sigma(p^\sigma) = g(p)$ for all $p^\sigma \in P_{n-1}^\sigma$, and $\rho^{-1}g^\sigma \in \text{Hom}_R(P_{n-1}^\sigma, R)$. Let $p^\sigma \in P_{n-1}^\sigma$. Then

$$\begin{aligned} Td_n^\sigma(\rho^{-1}g^\sigma)(p^\sigma) &= (\rho^{-1}g^\sigma)d_n^\sigma(p^\sigma) \\ &= (\rho^{-1}g^\sigma)(d_n(p)) \\ &= \rho^{-1}(g(d_n(p))) \\ &= \sigma^{-1}(gd_n(p)) \\ &= f(p). \end{aligned}$$

Therefore $f \in \text{im}(Td_n^\sigma)$ and $\ker(\tilde{\theta}) \subseteq \text{im}(Td_n^\sigma)$.

Now suppose $h \in \text{im}(Td_n^\sigma)$. Then there exists $\alpha \in \text{Hom}_R(P_{n-1}^\sigma, R)$ such that $Td_n^\sigma \alpha = h$; that is, $h = \alpha d_n^\sigma$. Let $p^\sigma \in P_n^\sigma$. Then

$$\begin{aligned} \theta_n(h)(p^\sigma) &= [\theta_n(\alpha d_n^\sigma)](p^\sigma) \\ &= \sigma[\alpha d_n^\sigma(p^\sigma)] \\ &= \sigma[\alpha d_n(p)] \\ &= \rho(\alpha(d_n(p))) \\ &= (\rho\alpha)d_n(p) \\ &= [Td_n(\rho\alpha)](p). \end{aligned}$$

Therefore $\theta_n(h) \in \text{im}(Td_n)$ and so $\theta_n(h) \in {}^\sigma[\text{im}(Td_n)]$.

Thus $\tilde{\theta}$ induces an isomorphism

$$\begin{aligned} \bar{\theta} : \text{Ext}_R^n(P_n^\sigma, R) &\rightarrow {}^\sigma[\text{Ext}_R^n(P_n, R)] \\ f + \text{im}(Td_n^\sigma) &\mapsto \theta_n(f) + {}^\sigma[\text{im}(Td_n)]. \end{aligned}$$

□

Now we are in a position to state and prove the required version of Theorem 5.1.3.

Proposition 5.1.7. *Let R be a ring and x be a normal non-zero divisor in R . Let $\sigma : R \rightarrow R$ be the automorphism obtained by conjugating by x ; that is, $xr = \sigma(r)x$. Let M be a right R -module and let*

$$\rho : M^\sigma \rightarrow M$$

be right multiplication by x .

Then the map ρ is an R -module homomorphism. Furthermore, from the exact sequence

$$0 \rightarrow M^\sigma \xrightarrow[\rho]{} M \rightarrow M/Mx \rightarrow 0,$$

we can construct a long exact sequence

$$\begin{aligned} \dots \rightarrow \text{Ext}_R^n(M/Mx, R) \rightarrow \text{Ext}_R^n(M, R) \xrightarrow{\psi} \\ {}^\sigma[\text{Ext}_R^n(M, R)] \rightarrow \text{Ext}_R^{n+1}(M/Mx, R) \rightarrow \dots, \end{aligned}$$

where ψ is left multiplication by x .

Proof. That ρ is an R -module homomorphism is given in Lemma 1.3.12.

Let $\dots \rightarrow P_1 \xrightarrow{d_1} P_0 \xrightarrow{\epsilon} M \rightarrow 0$ be a projective resolution for M . Then by Lemma 5.1.4, $\dots \rightarrow P_1^\sigma \xrightarrow{d_1^\sigma} P_0^\sigma \xrightarrow{\epsilon^\sigma} M^\sigma \rightarrow 0$ is a projective resolution for M^σ .

Consider the diagram

$$\begin{array}{ccccccc} \dots & \xrightarrow{d_2^\sigma} & P_1^\sigma & \xrightarrow{d_1^\sigma} & P_0^\sigma & \xrightarrow{\epsilon^\sigma} & M^\sigma \longrightarrow 0 \\ & & & & & & \downarrow \rho \\ \dots & \xrightarrow{d_2} & P_1 & \xrightarrow{d_1} & P_0 & \xrightarrow{\epsilon} & M \longrightarrow 0. \end{array}$$

By the Comparison Theorem ([37] Theorem 6.9), there exists a chain map g over ρ , where $g = \{g_n : P_n^\sigma \rightarrow P_n\}$ and any chain map will result in the same ρ^* . Define $g_n : P_n^\sigma \rightarrow P_n$ to be right multiplication by x for all $n \geq 0$. It is easy to check that this defines a chain map over ρ , and we have the following commutative diagram:

$$\begin{array}{ccccccc} \dots & \xrightarrow{d_2^\sigma} & P_1^\sigma & \xrightarrow{d_1^\sigma} & P_0^\sigma & \xrightarrow{\epsilon^\sigma} & M^\sigma \longrightarrow 0 \\ & & \downarrow g_1 & & \downarrow g_0 & & \downarrow \rho \\ \dots & \xrightarrow{d_2} & P_1 & \xrightarrow{d_1} & P_0 & \xrightarrow{\epsilon} & M \longrightarrow 0. \end{array}$$

Apply the contravariant functor $T := \text{Hom}_R(-, R)$,

$$\begin{array}{ccccccc} 0 & \longrightarrow & \text{Hom}_R(M^\sigma, R) & \longrightarrow & \text{Hom}_R(P_0^\sigma, R) & \xrightarrow{Td_1^\sigma} & \text{Hom}_R(P_1^\sigma, R) \longrightarrow \dots \\ & & \uparrow \rho^* & & \uparrow Tg_0 & & \uparrow Tg_1 \\ 0 & \longrightarrow & \text{Hom}_R(M, R) & \longrightarrow & \text{Hom}_R(P_0, R) & \xrightarrow{Td_1} & \text{Hom}_R(P_1, R) \longrightarrow \dots \end{array}$$

Consider $T\rho$.

By definition, $T\rho(f) = f \cdot \rho$ for all $f \in \text{Hom}_R(M, R)$. Let $m^\sigma \in M^\sigma$. Then

$$\begin{aligned} [T\rho(f)](m^\sigma) &= [f \cdot \rho](m^\sigma) \\ &= f(\rho(m^\sigma)) \\ &= f(mx) \\ &= f(m)x \quad (\text{usual multiplication in the ring } R). \end{aligned}$$

Thus

$$\begin{aligned} Tg_i : \text{Hom}_R(P_i, R) &\rightarrow \text{Hom}_R(P_i^\sigma, R) \\ f &\mapsto \{f \cdot g_i : p \mapsto f(p)x\} \end{aligned} \quad (\dagger)$$

for all $i \geq 0$. Now, $\text{Ext}_R^n(M, R) = \ker(Td_{n+1}) / \text{im}(Td_n)$, so let $z_{n+1} + \text{im}(Td_n) \in \text{Ext}_R^n(M, R)$, where $z_{n+1} \in \ker(Td_{n+1}) \subseteq \text{Hom}_R(P_n, R)$. Recall that, by definition (Definition 1.5.7),

$$\rho^* : \text{Ext}_R^n(M, R) \rightarrow \text{Ext}_R^n(M^\sigma, R)$$

is given by

$$\rho^* (z_{n+1} + \text{im}(Td_n)) = Tg_n z_{n+1} + \text{im}(Td_n^\sigma),$$

and we have an exact sequence (Theorem 1.5.8)

$$\begin{aligned} \dots \rightarrow \text{Ext}_R^n(M/Mx, R) \rightarrow \text{Ext}_R^n(M, R) \xrightarrow{\rho^*} \\ \text{Ext}_R^n(M^\sigma, R) \rightarrow \text{Ext}_R^{n+1}(M/Mx, R) \rightarrow \dots \end{aligned}$$

Recall the isomorphism

$$\begin{aligned} \bar{\theta} : \text{Ext}_R^n(P_n^\sigma, R) &\rightarrow \sigma[\text{Ext}_R^n(P_n, R)] \\ f + \text{im}(Td_n^\sigma) &\mapsto \theta_n(f) + \sigma[\text{im}(Td_n)] \end{aligned}$$

from Lemma 5.1.6. Then

$$\bar{\theta}(Tg_n z_{n+1} + \text{im}(Td_n^\sigma)) = \theta_n(Tg_n z_{n+1}) + \sigma[\text{im}(Td_n)].$$

Now we consider what $\theta_n(Tg_n z_{n+1})$ does to an element of P_n . Let $p \in P_n$. Then

$$\begin{aligned} \theta_n(Tg_n z_{n+1})(p) &= \sigma(Tg_n(z_{n+1})(p)) \\ &= \sigma(z_{n+1}(p)x) \quad \text{from } (\dagger) \\ &= \sigma(z_{n+1}(p))x \\ &= xz_{n+1}(p), \quad \text{since } xr = \sigma(r)x \text{ for } r \in R. \end{aligned}$$

Thus (by the left module action on $\sigma[\text{Hom}_R(M, R)]$),

$$\theta_n Tg_n(z_{n+1})(p) = (x.z_{n+1})(p)$$

for all $p \in P_n$. Thus $\theta_n Tg_n$ multiplies an element of $\sigma[\text{Hom}_R(M, R)]$ on the left by x . Therefore

$$\begin{aligned} \bar{\theta}(Tg_n z_{n+1} + \text{im}(Td_n^\sigma)) &= \theta_n(Tg_n z_{n+1}) + \sigma[\text{im}(Td_n)] \\ &= x.z_{n+1} + \sigma[\text{im}(Td_n)] \\ &= x(z_{n+1} + \sigma[\text{im}(Td_n)]), \end{aligned}$$

and we have a long exact sequence

$$\begin{aligned} \dots \rightarrow \text{Ext}_R^n(M/Mx, R) \rightarrow \text{Ext}_R^n(M, R) \xrightarrow{\psi} \\ \sigma[\text{Ext}_R^n(M, R)] \rightarrow \text{Ext}_R^{n+1}(M/Mx, R) \rightarrow \dots, \end{aligned}$$

where $\psi = \bar{\theta}\rho^*$ and is therefore left multiplication by x . □

Lemma 5.1.8. *Let R be a noetherian ring and let M be a finitely generated right R -module. Then $\text{Ext}_R^i(M, R)$ is noetherian, for all i .*

Proof. We can construct a free resolution of M

$$\dots \rightarrow R^{n_2} \xrightarrow{d_2} R^{n_1} \xrightarrow{d_1} R^{n_0} \rightarrow M \rightarrow 0,$$

and apply $T := \text{Hom}_R(-, R)$ to the deleted resolution to obtain the exact sequence

$$0 \rightarrow \text{Hom}_R(R^{n_0}, R) \xrightarrow{Td_1} \text{Hom}_R(R^{n_1}, R) \xrightarrow{Td_2} \text{Hom}_R(R^{n_2}, R) \rightarrow \dots$$

Then $\text{Ext}_R^i(M, R) = \ker(Td_{i+1})/\text{im}(Td_i)$. Now,

$$\ker Td_{i+1} \subseteq \text{Hom}_R(R^{n_i}, R) \cong \underbrace{\oplus \text{Hom}_R(R, R)}_{n \text{ copies}} \cong \underbrace{\oplus R}_{n \text{ copies}}.$$

Therefore $\text{Hom}_R(R^{n_i}, R)$ is a noetherian R -module, and thus $\ker(Td_{i+1})$ is also noetherian and so is $\text{Ext}_R^i(M, R)$. \square

Lemma 5.1.9. *(cf [29] Lemma 2.1) Let R be a noetherian ring and $x \in R$ be a normal element. If M is a finitely generated x -torsion free right R -module such that M/Mx is a projective right R/Rx -module, and $M_x := M \otimes_R R_x$ is a projective right R_x -module, then M is a projective right R -module.*

Proof. From [3] Proposition 1.6, we have

$$R_x \otimes_R \text{Ext}_R^i(M, R) \cong \text{Ext}_{R_x}^i(M \otimes_R R_x, R \otimes_R R_x).$$

Now, $\text{Ext}_{R_x}^i(M \otimes_R R_x, R \otimes_R R_x) = 0$ for $i \geq 1$ (M_x being a projective R_x -module). Thus, from [33] Proposition 2.1.17(iii), $\text{Ext}_R^i(M, R)$ is x -torsion for $i \geq 1$.

Let $\sigma : R \rightarrow R$ be the automorphism induced on R by conjugation by x ; that is, $xr = \sigma(r)x$, and recall the definition of M^σ . Then we have an R -module map

$$f : M^\sigma \rightarrow Mx \text{ defined by } m^\sigma \mapsto mx$$

and since M is x -torsion free, f is an isomorphism. We have the following exact sequence of R -modules:

$$0 \rightarrow M^\sigma \xrightarrow[\rho]{} M \rightarrow M/Mx \rightarrow 0, \quad (\dagger)$$

where ρ is right multiplication by x .

By applying $\text{Ext}_R(-, R)$ to the exact sequence (\dagger) and by using Proposition 5.1.7, we get the long exact sequence

$$\begin{aligned} \dots \rightarrow \text{Ext}_R^2(M/Mx, R) \rightarrow \text{Ext}_R^2(M, R) \xrightarrow{\psi} \\ \sigma[\text{Ext}_R^2(M, R)] \rightarrow \text{Ext}_R^3(M/Mx, R) \rightarrow \dots, \end{aligned}$$

where ψ is left multiplication by x . Now, $\text{pd}_{R/xR}(M/Mx) = 0$, and thus by [33] Theorem 7.3.5, $\text{pd}_R(M/Mx) = 1$. Therefore $\text{Ext}_R^2(M/Mx, R) = 0 = \text{Ext}_R^3(M/Mx, R)$ and

$$\text{Ext}_R^2(M, R) \xrightarrow[\psi]{} \sigma[\text{Ext}_R^2(M, R)]$$

is an isomorphism. Therefore $\sigma[\text{Ext}_R^2(M, R)] \cong x\text{Ext}_R^2(M, R)$. Also, $\sigma[\text{Ext}_R^2(M, R)] \cong_{\mathbb{Z}} \text{Ext}_R^2(M, R)$ as abelian groups, and so

$$\text{Ext}_R^2(M, R) \cong_{\mathbb{Z}} x\text{Ext}_R^2(M, R)$$

as abelian groups. We claim that $\text{Ext}_R^2(M, R) = 0$.

Let $E = \text{Ext}_R^2(M, R)$ and let $K_i = \{e \in E \mid x^i e = 0\}$. The K_i are submodules of E and $0 \leq K_1 \leq K_2 \leq \dots \leq E$. Now, by Lemma 5.1.8, E is noetherian and this chain must stop, at K_n , say. However, E is also x -torsion and therefore $E = \cup K_i$, so $E = K_n$ and $x^n E = 0$. Also,

$$\text{Ext}_R^2(M, R) \cong_{\mathbb{Z}} x\text{Ext}_R^2(M, R) \cong_{\mathbb{Z}} \dots \cong_{\mathbb{Z}} x^n \text{Ext}_R^2(M, R) = 0;$$

that is, $\text{Ext}_R^2(M, R) = 0$, as required.

Therefore, by [5] Exercise 6.9, $\text{Ext}_R^2(M, N) = 0$ for all R -modules N and so $\text{pd}_R(M) \leq 1$.

Now consider the exact sequence of R -modules:

$$\text{Ext}_R^1(M, R) \xrightarrow[\psi]{} \sigma[\text{Ext}_R^1(M, R)] \rightarrow \text{Ext}_R^2(M/Mx, R) = 0.$$

Then ψ is left multiplication by x and is surjective. Therefore

$$x\text{Ext}_R^1(M, R) \cong \sigma[\text{Ext}_R^1(M, R)] \cong_{\mathbb{Z}} \text{Ext}_R^1(M, R)$$

and by the same argument as before, $\text{Ext}_R^1(M, N) = 0$ for all R -modules N . Thus $\text{Ext}_R^1(M, R) = 0$ and $\text{pd}_R(M) < 1$; that is, $\text{pd}_R(M) = 0$. \square

Rather than hide the inductive step mentioned earlier in the proof of the main result, we choose to explicitly state and prove it as a separate proposition.

Proposition 5.1.10. *Let R be a noetherian ring and let $x \in R$ be a normal element. If M is a finitely generated x -torsionfree right R -module such that $\text{pd}_{R/Rx}(M/Mx) \leq n$ and $\text{pd}_{R_x}(M_x) \leq n$, then $\text{pd}_R(M) \leq n$.*

Proof. The proof is by induction on n . Lemma 5.1.9 implies that the result is true for $n = 0$.

Suppose $n > 0$. We have the following sequence of R -modules

$$0 \rightarrow K \rightarrow F \xrightarrow{\theta} M \rightarrow 0, \quad (\dagger)$$

where F is a free R -module and $K = \ker \theta$.

Claim 1: $\text{pd}_R(M) \leq n \Rightarrow \text{pd}_R(K) \leq n - 1$.

Proof of claim: Suppose that $\text{pd}_R(M) \leq n$. For any R -module B , apply $\text{Ext}_R(-, B)$ to the exact sequence (\dagger) to obtain the exact sequence

$$\dots \rightarrow \text{Ext}_R^n(F, B) \rightarrow \text{Ext}_R^n(K, B) \rightarrow \text{Ext}_R^{n+1}(M, B) \rightarrow \dots$$

Then, since $\text{Ext}_R^n(F, B) = 0 = \text{Ext}_R^{n+1}(M, B)$, we have $\text{Ext}_R^n(K, B) = 0$ for all R -modules B ; that is, $\text{pd}_R(K) \leq n - 1$.

Now, θ induces a map $\bar{\theta} : F/Fx \rightarrow M/Mx$ defined by $\bar{\theta}(\bar{f}) = \bar{\theta}(f + Fx) := \theta(f) + Mx$ for $f \in F$. It is easy to check that $\bar{\theta}$ is a well defined homomorphism of R/xR -modules.

Claim 2: $\ker(\bar{\theta}) \cong K/Kx$.

Proof of Claim 2: We will show that $\ker(\bar{\theta}) = (K + Fx)/Fx \cong K/Kx$.

Now, $\ker(\bar{\theta}) = \{\bar{f} \in F/Fx \mid \bar{\theta}(\bar{f}) = 0\} = \{f + Fx \in F/Fx \mid \theta(f) \in Mx\}$. So clearly,

$$(K + Fx)/Fx \subseteq \ker(\bar{\theta}),$$

since $\theta(K) = 0$. Suppose $\bar{f} \in \ker(\bar{\theta})$. Then $\theta(f) \in Mx = \theta(Fx)$, so $\theta(f) = \theta(f'x)$ for some $f' \in F$. Then $\theta(f - f'x) = 0$ and $f - f'x \in \ker(\theta) = K$. Therefore $f \in K + Fx$ and $\bar{f} \in (K + Fx)/Fx$. Thus, $\ker(\bar{\theta}) = (K + Fx)/Fx$.

Clearly $\ker(\bar{\theta}) = (K + Fx)/Fx \cong K/(K \cap Fx)$ and $Kx \subseteq K \cap Fx$. Now suppose $k \in K \cap Fx$. Then $k = gx$, for some $g \in F$ and $(g + K)x = 0$ in F/K . However, $F/K \cong M$ and is therefore x -torsion free; that is, $g + K = 0$, so $g \in K$ and $K \cap Fx = Kx$.

Therefore $\ker \bar{\theta} \cong K/Kx$, as required.

Thus we have the following exact sequence of R/xR -modules:

$$0 \rightarrow K/Kx \rightarrow F/Fx \xrightarrow{\bar{\theta}} M/Mx \rightarrow 0,$$

where F/Fx is a free R/Rx -module.

By hypothesis we have $\text{pd}_{R/Rx}(M/Mx) \leq n$. Therefore, by Claim 1, we have $\text{pd}_{R/Rx}(K/Kx) \leq n - 1$.

Also, R_x is flat as a left and right R -module, so from the exact sequence of R -modules (†) we obtain the exact sequence of R_x -modules

$$0 \rightarrow K \otimes_R R_x \rightarrow F \otimes_R R_x \rightarrow M \otimes_R R_x \rightarrow 0,$$

with $F \otimes_R R_x$ a free R_x -module. Therefore, as above, $\text{pd}(K \otimes_R R_x) \leq n - 1$.

Therefore, by the induction hypothesis, $\text{pd}(K) \leq n - 1$.

Apply $\text{Ext}_R(-, B)$, where B is some R -module, to the exact sequence (†) to obtain the exact sequence

$$\dots \rightarrow \text{Ext}_R^n(K, B) \rightarrow \text{Ext}_R^{n+1}(M, B) \rightarrow \text{Ext}_R^{n+1}(K, B) \rightarrow 0.$$

However, $\text{Ext}_R^n(K, B) = 0$ and $\text{Ext}_R^{n+1}(K, B) = 0$. Therefore $\text{Ext}_R^{n+1}(M, B) = 0$ and thus, $\text{pd}_R(M) \leq n$ as required. \square

Theorem 5.1.11. (cf [29] Lemma 2.2.) *Let x be a regular normal element of the noetherian ring R . Then $\text{gldim}(R) \leq \max\{1 + \text{gldim}(R/Rx), \text{gldim}(R_x)\}$. In fact, if $\text{gldim}(R/Rx)$ is finite, then we have equality*

$$\text{gldim} R = \max\{1 + \text{gldim}(R/Rx), \text{gldim}(R_x)\}.$$

Proof. If $\max\{1 + \text{gldim}(R/Rx), \text{gldim}(R_x)\} = \infty$, then there is nothing to prove. We may assume that $\max\{1 + \text{gldim}(R/Rx), \text{gldim}(R_x)\} = n \leq \infty$.

Let M be a finitely generated right R -module and let M_t denote the x -torsion submodule of M ; that is, $M_t := \{m \in M \mid mx^i = 0 \text{ for some } i\}$. Then we have an exact sequence of R -modules

$$0 \rightarrow M_t \rightarrow M \rightarrow M/M_t \rightarrow 0. \quad (*)$$

Now $\text{pd}_R(M) \leq \max\{\text{pd}_R(M_t), \text{pd}_R(M/M_t)\}$ (see, for example [37] Lemma 9.26). We will show that $\text{pd}_R(M_t) \leq n$ and $\text{pd}_R(M/M_t) \leq n$.

Since R is noetherian there exists an integer $\omega \in \mathbb{N}$ such that $M_t x^\omega = 0$.

We apply induction on ω to arrive at

$$\text{pd}_R(M_t) \leq 1 + \text{gldim}(R/Rx) \leq n.$$

Suppose that $\omega = 1$. Then $M_t x = 0$ and M_t is a non-zero R/Rx -module. Therefore, by [33] Theorem 7.3.5, if $\text{pd}_{R/Rx}(M_t) = m \leq \infty$, then $\text{pd}_R(M_t) = m + 1$. Therefore

$$\text{pd}_R(M_t) = \text{pd}_{R/Rx}(M_t) + 1 \leq \text{gldim}(R/Rx) + 1.$$

Now suppose $\omega > 1$ and consider the exact sequence of R -modules

$$0 \rightarrow M_t x \rightarrow M_t \rightarrow M_t/M_t x \rightarrow 0.$$

Then $\text{pd}_R(M_t) \leq \max\{\text{pd}_R(M_t x), \text{pd}_R(M_t/M_t x)\}$ and by the induction hypothesis $\text{pd}_R(M_t x) \leq n$ and $\text{pd}_R(M_t/M_t x) \leq n$. Therefore $\text{pd}_R(M_t) \leq n$, as required.

Now consider the x -torsion free module $N := M/M_t$. Then, since $\text{gldim}(R/Rx) \leq n$ and $\text{gldim}(R_x) \leq n$, we have

$$\text{pd}_{R/Rx}(N/Nx) \leq n \quad \text{and} \quad \text{pd}_{R_x}(N_x) \leq n.$$

Thus, by Proposition 5.1.10, we have $\text{pd}_R(M/M_t) \leq n$. Therefore

$$\text{gldim}(R) \leq \max\{1 + \text{gldim}(R/Rx), \text{gldim}(R_x)\}.$$

It only remains to obtain equality. Let \overline{M} be a non-zero R/Rx -module. Then by [33] Theorem 7.3.5, $\text{pd}_R(\overline{M}) = \text{pd}_{R/Rx}(\overline{M}) + 1$ and we obtain

$$\text{gldim}(R) \geq 1 + \text{gldim}(R/Rx)$$

and $\text{gldim}(R) \geq \text{gldim}(R_x)$ gives the required equality. \square

We obtain the immediate corollary.

Corollary 5.1.12. *Let $R = \bigoplus_{n \geq 0} R_n$ be a finitely generated \mathbb{N} -graded k -algebra with $\dim_k R_0 < \infty$. Let R be noetherian and $x \in R_d$, $d > 0$, be a normal non-zero divisor in R . If R/Rx and R_x are Auslander Regular rings, then R is Auslander Regular.*

Proof. From [27] Theorem 5.10 we have that if R/Rx is Auslander Gorenstein, then R is Auslander Gorenstein. It remains to show that R has finite global dimension; but this follows immediately from 5.1.11 and we have the result. \square

5.2 The Auslander Regularity of (non-graded) Rings

The purpose of the following work is to remove the graded condition from Corollary 5.1.12. These results are derived from results of Van Oystaeyen and Li Huishi which appear in [31] Chapter III.

Definition 5.2.1. Let R be a ring and M be a right R -module. Let

$$J_R(M) = \inf\{j_R(N) \mid N \subseteq M \text{ an } R\text{-submodule}\}$$

and let

$$\mathcal{C}_n^R = \{M \in R\text{-Mod} \mid J_R(M) \geq n\}.$$

Now consider the following exact sequence of R -modules

$$0 \rightarrow M_1 \rightarrow M \xrightarrow{\alpha} M_2 \rightarrow 0.$$

We have the following lemma from [31], Chapter III, Lemma 3.1.1.

Lemma 5.2.2. *With notation as above:*

- (i) \mathcal{C}_n^R is closed under: taking submodules, quotients, extension and direct limits;
- (ii) Suppose $M_1 \in \mathcal{C}_{n_1}^R$, $M_2 \in \mathcal{C}_{n_2}^R$ and $M \in \mathcal{C}_n^R$. Let $s = \min\{n_1, n_2\}$ and $t = \min\{n_1, n\}$. Then $M \in \mathcal{C}_s^R$ and $M_2 \in \mathcal{C}_t^R$;
- (iii) if $J_R(\text{Ext}_R^i(M_1, R)) \geq i$ and $J_R(\text{Ext}_R^i(M_2, R)) \geq i$ for all $i \geq 0$, then $J_R(\text{Ext}_R^i(M, R)) \geq i$ for all $i \geq 0$.

Clearly, if $J_R(\text{Ext}_R^i(M, R)) \geq i$ for all $i \geq 0$, then M satisfies the Auslander Condition.

Lemma 5.2.3. *Let R be a ring, $\sigma : R \rightarrow R$ be an automorphism of R and M be a right R -module. If $M \in \mathcal{C}_n^R$, then $M^\sigma \in \mathcal{C}_n^R$.*

Proof. If $M \in \mathcal{C}_n^R$, then $J_R(M) \geq n$. Let $N^\sigma \subseteq M^\sigma$ (so $N \subseteq M$). We must show that $j_R(N^\sigma) \geq n$. Certainly $j_R(N) \geq n$ and so

$$\begin{aligned} \text{Ext}_R^i(N, R) = 0 \text{ for all } i < n &\Rightarrow \sigma[\text{Ext}_R^i(N, R)] = 0 \text{ for all } i < n \\ &\Rightarrow \text{Ext}_R^i(N^\sigma, R) = 0 \text{ for all } i < n \\ &\Rightarrow j_R(N^\sigma) \geq n, \end{aligned}$$

as required. Therefore $M^\sigma \in \mathcal{C}_n^R$. □

In the following results, R is a right and left noetherian ring and x is a normal non-zero divisor in R .

Lemma 5.2.4. *Suppose we have an exact sequence of finitely generated right R -modules*

$$0 \rightarrow M_1 \rightarrow M^\sigma \xrightarrow{\mu} M \rightarrow M_2 \rightarrow 0,$$

where μ is right multiplication by x . Then

(i) if N is a submodule of M such that $Nx^t = 0$ for some t , then N has a finite filtration such that the corresponding i th subquotients are isomorphic to submodules of $M_1^{\sigma^{-i}}$;

(ii) if Q is a quotient module of M such that $Qx^t = 0$ for some t , then Q has a finite filtration such that the corresponding i th subquotients are isomorphic to quotients of $M_2^{\sigma^{-i}}$.

Proof.

(i) Define $N_i = \{n \in N \mid nx^i = 0\}$, then we have a finite filtration of N

$$0 = N_0 \subseteq N_1 \subseteq \dots \subseteq N_t = N.$$

Consider the R -module homomorphism

$$\begin{aligned} \phi : N_i &\rightarrow N_1^{\sigma^{-i+1}} \\ n_i &\mapsto n_i x^{i-1}. \end{aligned}$$

Then $\ker \phi = \{n_i \in N_i \mid n_i x^{i-1} = 0\} = N_{i-1}$. Also, $\text{im} \phi \subseteq N_1^{\sigma^{-i+1}}$ and $N_1^{\sigma} \subseteq \ker \mu \cong M_1$. Therefore

$$(N_i/N_{i-1})^{\sigma^i} \hookrightarrow N_1^{\sigma} \subseteq M_1,$$

and $N_i/N_{i-1} \hookrightarrow M_1^{\sigma^{-i}}$.

(ii) Define $Q_i = Qx^i$. Then we have a finite filtration of Q

$$Q = Q_0 \supseteq Q_1 \supseteq \dots \supseteq Q_t = 0.$$

Consider the homomorphism

$$\begin{aligned} \eta : (M/Mx)^{\sigma^{-i}} &\rightarrow Q_i/Q_{i+1} \\ m + Mx &\mapsto mx^i + Q_{i+1}. \end{aligned}$$

Then η is surjective and Q_i/Q_{i+1} is isomorphic to a quotient module of $(M/Mx)^{\sigma^i}$. □

Lemma 5.2.5. Suppose we have the following exact sequence of finitely generated R -modules:

$$0 \rightarrow M_1 \rightarrow M^{\sigma} \xrightarrow{\mu} M \rightarrow M_2 \rightarrow 0, \quad (\dagger)$$

where μ is right multiplication by x , such that

(i) $M_1 \in \mathcal{C}_n^R$;

(ii) $M_x \in \mathcal{C}_n^{R_x}$;

(iii) $M_2 \in \mathcal{C}_{n+1}^R$.

Then $M \in \mathcal{C}_n^R$.

Proof. First suppose that M is x -torsion free and notice that in this case $M_1 = 0$. Let N be a submodule of M . Then we must show that $j_R(N) \geq n$. We split the proof of this into two cases.

Case 1. M/N is an x -torsion free R -module.

Note that $N \cap Mx = Nx$, and thus N/Nx is isomorphic to a submodule of $M/Mx \cong M_2 \in \mathcal{C}_{n+1}^R$. Therefore $j_R(N/Nx) \geq n+1$.

Note also that $N_x \subseteq M_x \in \mathcal{C}_n^{R_x}$, thus $j_{R_x}(N_x) \geq n$ and $\text{Ext}_{R_x}^i(N_x, R_x) = 0$ for all $i \leq n-1$. Now, by [3] Proposition 1.6

$$\text{Ext}_{R_x}^i(N_x, R_x) \cong R_x \otimes_R \text{Ext}_R^i(N, R),$$

and so (by [33] Proposition 2.1.17) $\text{Ext}_R^j(N, R)$ is x -torsion for all $j \leq n-1$.

Consider the exact sequence

$$0 \rightarrow N^\sigma \rightarrow N \rightarrow N/Nx \rightarrow 0.$$

Apply $\text{Ext}_R(-, R)$ to obtain the long exact sequence

$$\begin{aligned} \dots \rightarrow \text{Ext}_R^i(N/Nx, R) \rightarrow \text{Ext}_R^i(N, R) \xrightarrow{\psi} \\ \sigma[\text{Ext}_R^i(N, R)] \rightarrow \text{Ext}_R^{i+1}(N/Nx, R) \rightarrow \dots, \end{aligned}$$

where ψ is left multiplication by x . Now, if $i \leq n-1$, then $\text{Ext}_R^i(N/Nx, R) = 0 = \text{Ext}_R^{i+1}(N/Nx, R)$ and

$$\text{Ext}_R^i(N, R) \xrightarrow[\psi]{} \sigma[\text{Ext}_R^i(N, R)]$$

is an isomorphism. Then by an exactly similar argument to that in the proof of Lemma 5.1.9, $\text{Ext}_R^i(N, R) = 0$ for all $i \leq n-1$; that is, $j_R(N) \geq n$.

Case 2. M/N has a nonzero x -torsion module.

Let $T := \{m \in M \mid mx^i \in N \text{ some } i\}$. Then T is an R -submodule of M . Consider M/T and suppose that $(m+T)x = 0$ for some $m \in M$. Then $mx \in T$, and so $m \in T$. Therefore M/T is x -torsion free, and by Case 1 $j_R(T) \geq n$. Also, since $T \cap Mx = Tx$, we have that T/Tx is isomorphic to a submodule of $M/Mx \in \mathcal{C}_{n+1}^R$. Therefore $j_R(T/Tx) \geq n+1$.

Now consider T/N . Since R is noetherian there exists $t > 0$ such that $(T/N)x^t = 0$. Thus we can apply Lemma 5.2.4(ii) with $M = T$ and $Q = T/N$. Then Q has a finite filtration such that the i th quotient is isomorphic to a quotient of $(T/Tx)^{\sigma^{-i}} \in \mathcal{C}_{n+1}^R$. Therefore $T/N \in \mathcal{C}_{n+1}^R$.

Consider the exact sequence

$$0 \rightarrow N \rightarrow T \rightarrow T/N \rightarrow 0$$

and apply $\text{Ext}_R(-, R)$ to obtain the long exact sequence

$$\dots \rightarrow \text{Ext}_R^i(T, R) \rightarrow \text{Ext}_R^i(N, R) \rightarrow \text{Ext}_R^{i+1}(T/N, R) \rightarrow \dots$$

If $i < n$, then $\text{Ext}_R^i(T, R) = 0 = \text{Ext}_R^{i+1}(T/N, R)$ and thus $\text{Ext}_R^i(N, R) = 0$ for all $i < n$; that is, $j_R(N) \geq n$.

Therefore, the result holds if M is x -torsion free. Now suppose that M has a nonzero torsion module $T' := \{m \in M \mid mx^i = 0 \text{ for some } i\}$, and consider the exact sequence

$$0 \rightarrow T' \rightarrow M \rightarrow M/T' \rightarrow 0.$$

We will show that $T' \in \mathcal{C}_n^R$ and $M/T' \in \mathcal{C}_n^R$. First consider T' . Since R is noetherian there exists $t > 0$ such that $T'x^t = 0$. Therefore, by Lemma 5.2.4(i), T' has a finite filtration such that the i th quotient module is isomorphic to a submodule of $M_1^{\sigma^{-i}} \in \mathcal{C}_n^R$. Since \mathcal{C}_n^R is closed under taking submodules and extensions, we have $T' \in \mathcal{C}_n^R$. Now consider M/T' . From the original exact sequence (†) we get the exact sequence

$$0 \rightarrow \left(\frac{M}{T'}\right)^\sigma \rightarrow \frac{M}{T'} \rightarrow \frac{M}{Mx + T'} \rightarrow 0,$$

which satisfies the criteria required for our result (with M/T' playing the part of M). Therefore, since M/T' is x -torsion free, we have that $M/T' \in \mathcal{C}_n^R$. It follows from Lemma 5.2.2 that $M \in \mathcal{C}_n^R$. \square

Lemma 5.2.6. *Let R be a noetherian ring and $x \in R$ be a regular normal element. Let M be a right R/Rx -module. If M satisfies the Auslander Condition as an R/Rx -module, then M satisfies the Auslander Condition as an R -module.*

Proof. Let $f \in \text{Hom}_R(M, R)$. Then $f(m)x = f(mx) = f(0) = 0$. However, x is regular in R , so $f(m) = 0$ and thus $f = 0$. Therefore $\text{Ext}_R^0(M, R) \cong \text{Hom}_R(M, R) = 0$ and so we consider $\text{Ext}_R^i(M, R)$, where $i \geq 1$. By [27] Remark 3.4, we have

$$\text{Ext}_R^{i+1}(M, R) \cong \text{Ext}_{R/Rx}^i(R, R/Rx) \text{ for } i > 0.$$

Let $i \geq 1$ and $N \subseteq \text{Ext}_R^i(M, R)$. We must show that $j_R(N) \geq i$. By above, N is isomorphic to an R -submodule N' of $\text{Ext}_{R/Rx}^{i-1}(M, R/Rx)$. By hypothesis, $j_{R/Rx}(N') \geq i-1$ therefore, by [33] Theorem 7.3.5, $j_R(N') \geq (i-1) + 1 = i$ and thus $j_R(N) \geq i$ as required. \square

Lemma 5.2.7. *Let M be a finitely generated x -torsion free right R -module. Suppose M_x satisfies the Auslander Condition as an R_x -module and M/Mx satisfies the Auslander Condition as an R/Rx -module. Then M satisfies the Auslander Condition as an R -module.*

Proof. Consider the exact sequence

$$0 \rightarrow M^\sigma \xrightarrow[\mu]{} M \rightarrow M/Mx \rightarrow 0,$$

where μ is multiplication by x and apply $\text{Ext}_R(-, R)$ to obtain the long exact sequence

$$\begin{aligned} \dots \rightarrow \text{Ext}_R^n(M/Mx, R) \rightarrow \text{Ext}_R^n(M, R) \xrightarrow{\psi} \\ \sigma[\text{Ext}_R^n(M, R)] \rightarrow \text{Ext}_R^{n+1}(M/Mx, R) \rightarrow \dots \end{aligned}$$

So we have an exact sequence

$$0 \rightarrow M_1 \rightarrow \text{Ext}_R^n(M, R) \xrightarrow[\psi]{} \sigma[\text{Ext}_R^n(M, R)] \rightarrow M_2 \rightarrow 0$$

where M_1 is a quotient module of $\text{Ext}_R^n(M/Mx, R)$ and M_2 is a subfactor of $\text{Ext}_R^{n+1}(M/Mx, R)$. By Lemma 5.2.6, M/Mx satisfies the Auslander Condition as an R -module. Therefore $M_1 \in \mathcal{C}_n^R$ and $M_2 \in \mathcal{C}_{n+1}^R$. Also, since M_x satisfies the Auslander Condition as an R_x -module,

$$[\text{Ext}_R^n(M, R)]_x \cong \text{Ext}_R^n(M, R) \otimes_R R_x \cong \text{Ext}_{R_x}^n(M_x, R_x) \in \mathcal{C}_n^{R_x}.$$

Now, from Proposition 5.1.7, ψ is left multiplication by x , thus by Lemma 5.2.5 for left modules, we have $\text{Ext}_R^n(M, R) \in \mathcal{C}_n^R$.

Theorem 5.2.8. *Let R be a noetherian ring and $x \in R$ be a regular normal element of R . If R_x and R/Rx are Auslander Regular, then R is Auslander Regular.*

Proof. By 5.1.11

$$\text{gldim}(R) \leq \max\{1 + \text{gldim}(R/Rx), \text{gldim}(R_x)\},$$

so R has finite global dimension. It remains to show that if M is a finitely generated R -module, then M satisfies the Auslander Condition. Let M be a finitely generated R -module and let $T := \{m \in M \mid mx^i = 0 \text{ some } i\}$. We have an exact sequence of finitely generated modules

$$0 \rightarrow T \rightarrow M \rightarrow M/T \rightarrow 0.$$

By [31] Chapter III, Lemma 2.1.4, if T and M/T satisfy the Auslander Condition as R -modules, then M satisfies the Auslander Condition as an R -module.

Consider $N := M/T$.

Claim: N satisfies the Auslander Condition as an R -module.

N is an x -torsion free finitely generated R -module and since R_x and R/Rx are Auslander Regular, Lemma 5.2.7 implies that N satisfies the Auslander Condition as an R -module.

Consider T .

Since R is noetherian there exists $t > 0$ such that $Tx^t = 0$. We use induction on t to show that T satisfies the Auslander Condition as an R -module. If $t = 1$ then T is an R/Rx -module which, by hypothesis, satisfies the the Auslander Condition as an R/Rx -module. Therefore by Lemma 5.2.6, T satisfies the Auslander Condition as an R -module. Now suppose $t > 1$. Then we have an exact sequence

$$0 \rightarrow Tx^{t-1} \rightarrow T \rightarrow T/Tx^{t-1} \rightarrow 0.$$

By the induction hypothesis, both Tx^{t-1} and T/Tx^{t-1} satisfy the Auslander Condition as R -modules. Therefore by [31] Chapter III, Lemma 2.1.4, T satisfies the Auslander Condition as an R -module.

Therefore M satisfies the Auslander Condition as an R -module and R is Auslander Regular. \square

Chapter 6

Krull Dimension of q -skew polynomial rings over a commutative ring

The concept of classical Krull dimension was first considered for commutative noetherian rings and was defined using (strictly decreasing) chains of prime ideals. Consider a chain $P_0 \supsetneq P_1 \supsetneq \dots \supsetneq P_n$ of prime ideals in a commutative noetherian ring R . We say that this chain has length n and define the **classical Krull dimension** of R to be the supremum of the lengths of all such chains. Notice that all commutative artinian rings have classical Krull dimension 0. Thus the classical Krull dimension can be seen as a measure of how far a ring is from being artinian. However, in noncommutative theory, the length of the chains of prime ideals is no longer necessarily indicative of this. In particular, we would have that all simple rings have Krull dimension 0, while not all simple rings are artinian. The definition of Krull dimension commonly used is due to Rentshler and Gabriel and is given by a transfinite induction. We introduce this definition in Section 6.1, where we also state some basic results needed in the rest of the chapter. One should be aware that this notion of Krull dimension is not always defined, but in the case of noetherian rings/modules there is no problem.

The main aim of this chapter is to study the Krull dimension of a particular type of skew polynomial ring, namely q -skew polynomial rings over a commutative ring of finite Krull dimension. A skew polynomial ring $R[x; \sigma, \delta]$ is called a **q -skew polynomial ring** if $\delta\sigma = q\sigma\delta$ for some scalar q . In general, calculations for a skew polynomial ring can quickly become unmanageable. However, in a q -skew polynomial ring the relationship between σ and δ enables us to handle calculations in a reasonable manner. This type of skew polynomial ring has been studied in [14], [20] and [38]. In particular, in [38], Woodward establishes some criteria on

the maximal ideals of a commutative ring R (of finite global dimension) which determine the global dimension of a q -skew polynomial ring extension of R . In Section 6.3 we will show that the same criteria will establish the Krull dimension of a q -skew polynomial ring extension of a commutative noetherian ring of finite Krull dimension. This obtains for us the final result: the global dimension of a q -skew polynomial ring extension of R is equal to the Krull dimension of R .

6.1 Krull Dimension

Standard references for basic results regarding Krull dimension are [23] Chapter 13 and [33] Chapter 6. We begin with the definition of the Krull dimension of a module.

Definition 6.1.1. Let R be a ring and M be an R -module. We define the **Krull dimension** of M , denoted $\text{Kdim}(M)$, by a transfinite induction. First define

$$\text{Kdim}(M) := -1 \text{ if and only if } M = 0.$$

Then consider an ordinal $\alpha \geq 0$ and assume that we have already defined what it means for a module to have Krull dimension β for all $\beta < \alpha$. Then M has Krull dimension α if

- (i) we have not already defined $\text{Kdim}(M) = \beta$ for some $\beta < \alpha$, and
- (ii) for every countable descending chain

$$M_0 \supseteq M_1 \supseteq M_2 \supseteq \dots$$

of submodules of M , we have $\text{Kdim}(M_i/M_{i+1}) < \alpha$ for all but finitely many indices i ; that is, for all but finitely many i the Krull dimension of M_i/M_{i+1} has already been defined to be an ordinal less than α .

It may be the case that $\text{Kdim}(M) = \alpha$ does not hold for any ordinal α and in this event we say that $\text{Kdim}(M)$ is not defined.

Given a ring R , the **right Krull dimension** of R is the Krull dimension of the ring when considered as a right module over itself and is sometimes denoted by $\text{rKdim}(R)$. Of course there is an equivalent definition on the left and it remains an open question as to whether or not these two values are equal. In this thesis, when referring to the Krull dimension of a ring, we will mean the right Krull dimension.

Example 6.1.2. The modules of Krull dimension 0 are precisely the artinian modules.

Example 6.1.3. The ring of integers \mathbb{Z} has Krull dimension 1, since \mathbb{Z} itself is not artinian, but all its factor rings are.

The following Lemma removes the need to worry about the existence of Krull dimension for a noetherian module.

Lemma 6.1.4. *If M is a noetherian module, then $\text{Kdim}(M)$ is defined.*

Proof. See [23] Lemma 13.3. □

Definition 6.1.5. Let $\alpha \geq 0$ be an ordinal. A module M is α -critical if $\text{Kdim}(M) = \alpha$ and $\text{Kdim}(M/N) < \alpha$ for all nonzero submodules N of M . A module is called **critical** if it is α -critical for some ordinal α .

Example 6.1.6. The 0-critical modules are precisely the simple modules.

Example 6.1.7. The ring of integers \mathbb{Z} is 1-critical as a module over itself (cf. Example 6.1.3).

Lemma 6.1.8. *Let M be an R -module and N be a nonzero submodule of M .*

(i) If M has Krull dimension, then M contains a critical submodule.

(ii) If M is α -critical, then N is α -critical. □

Lemma 6.1.9 is a well known result (see, for example, [33] Proposition 6.5.4) which will, not surprisingly, be of great use when we come to calculating the Krull dimension of q -skew polynomial rings (Section 6.3).

Lemma 6.1.9. *Let R be a noetherian ring, σ be an automorphism on R and δ be a σ -derivation. Then*

$$\text{Kdim}(R) \leq \text{Kdim}(R[x; \sigma, \delta]) \leq \text{Kdim}(R) + 1.$$

□

6.2 q -skew polynomial rings

Recall that a (left) skew derivation on a ring R is a pair (σ, δ) where $\sigma : R \rightarrow R$ is an automorphism of R and δ is a (left) σ -derivation on R ; that is, δ is an additive

map such that $\delta(rs) = \sigma(r)\delta(s) + \delta(r)s$ for all $r, s \in R$. We will say that $q \in R$ is a (σ, δ) -constant if $\sigma(q) = q$ and $\delta(q) = 0$. If q is a (σ, δ) -constant, then

$$\sigma(qr) = q\sigma(r) \text{ and } \delta(qr) = q\delta(r) \text{ for all } r \in R.$$

Definition 6.2.1. Let (σ, δ) be a skew derivation on R and let q be a central (σ, δ) -constant. If $\delta\sigma = q\sigma\delta$, then (σ, δ) is a q -skew derivation. If (σ, δ) is a q -skew derivation on R , then $R[x; \sigma, \delta]$ is called a q -skew polynomial ring.

If (σ, δ) is a q -skew derivation on R and $S := R[x; \sigma, \delta]$, then σ extends to an automorphism of S .

Lemma 6.2.2. Given a q -skew polynomial ring $S = R[x; \sigma, \delta]$, the automorphism σ extends to an automorphism of S via

$$\begin{aligned} \sigma : S &\rightarrow S \\ x &\mapsto q^{-1}x. \end{aligned}$$

Proof. See [20] Section 2.4(ii). □

In q -skew polynomial rings, the relation $\delta\sigma = q\sigma\delta$ allows us to derive q -Leibniz rules. First we familiarise ourselves with q -binomial coefficients.

Definition 6.2.3. For an indeterminate u and integers $n \geq m \geq 0$ we define u -binomial coefficients:

- (a) $(m)_u := u^{m-1} + u^{m-2} + \dots + 1$ for $m > 0$;
- (b) $(m!)_u := (m)_u (m-1)_u \dots (1)_u$ for $m > 0$ and $(0!)_u := 1$;
- (c) $\binom{n}{m}_u := (n!)_u / (m!)_u ((n-m)!)_u$.

If $m > n$, we define $\binom{n}{m}_u := 0$.

In fact u -binomial coefficients have many properties similar to those of the usual binomial coefficients.

Lemma 6.2.4. Let $n \geq m \geq 0$. Then

- (a) $\binom{n}{m}_u$ is a polynomial in u with non-negative coefficients;
- (b) $\binom{n}{0}_u = \binom{n}{n}_u = 1$;
- (c) $\binom{n}{m}_u = \binom{n-1}{m}_u + u^{n-m} \binom{n-1}{m-1}_u = \binom{n-1}{m-1}_u + u^m \binom{n-1}{m}_u$ for all $n > m \geq 0$.

Proof. See [1] Theorem 3.2. □

The q -binomial coefficient $\binom{n}{m}_q$ in R is defined as the evaluation of $\binom{n}{m}_u$ at $u = q$. Note that the 1-binomial coefficient is just the usual binomial coefficient; that is, $\binom{n}{m}_1 = \binom{n}{m}$. We can now write down the q -Leibniz rules for q -skew polynomial rings ([14] Lemma 6.2).

Lemma 6.2.5. *The q -Leibniz rules*

Let $R[x; \sigma, \delta]$ be a q -skew polynomial ring. Then

$$(a) \ x^n r = \sum_{i=0}^n \binom{n}{i}_q \sigma^{n-i} \delta^i(r) x^{n-i} \text{ for all } r \in R;$$

$$(b) \ \delta^n(rs) = \sum_{i=0}^n \binom{n}{i}_q \sigma^{n-i} \delta^i(r) \delta^{n-i}(s) \text{ for all } r, s \in R. \quad \square$$

6.3 The Krull dimension of q -skew polynomial rings

The remainder of the chapter is devoted to finding the Krull dimension of a q -skew polynomial ring S over a commutative noetherian ring R with finite Krull dimension. By Lemma 6.1.9

$$\text{Kdim}(R) \leq \text{Kdim}(S) \leq \text{Kdim}(R) + 1,$$

so there are only two possibilities for the Krull dimension of S . Goodearl and Lenagan, [16], have calculated the Krull dimension of a skew Laurent extension T over a commutative noetherian ring R with finite Krull dimension. When M is a critical noetherian R -module, the Krull dimension of $M \otimes_R T$ is described in terms of the Krull dimension of modules of the form $M' \otimes_R T$, where M' ranges over critical subfactors of M with Krull dimension less than M ([16], II). This expression is then used in an inductive procedure to calculate the Krull dimension of a module $N \otimes_R T$ (where N is an R -module) in terms of the Krull dimension of modules $M \otimes_R T$ as M ranges over the simple subfactors of N ([16], III). We aim to use this method to obtain an expression for the Krull dimension of the S -module $N \otimes_R S$, where N is a noetherian R -module with finite Krull dimension. This of course results in an expression for the Krull dimension of $R \otimes_R S \cong S$. We then manipulate this expression to establish a criterion on the maximal ideals of R which will determine which of the two possibilities holds for the Krull dimension of S .

A concept which will be useful is that of compressibility. An R -module M is **compressible** if for any nonzero submodule N of M , there is a monomorphism

$M \hookrightarrow N$. It is well known (and easy to see) that a compressible R -module with Krull dimension is critical (see, for example, [33] Lemma 6.9.4). In fact, later we will see that (when R is a commutative noetherian ring) an R -module M is compressible if and only if it is critical. As so often has been the case in this thesis, we must ‘twist’ the idea of compressibility.

Definition 6.3.1. Let R be a ring and let σ be an automorphism of R . Let M be a right R module. We say that M is σ -compressible if whenever N is a nonzero submodule of M , there exists an integer n such that $M^{\sigma^n} \hookrightarrow N$.

Remark. The lattice of submodules of M is exactly that of M^σ . Therefore

$$\text{Kdim}(M) = \text{Kdim}(M^\sigma)$$

and M is critical if and only if M^σ is critical.

Lemma 6.3.2. *Let M be a right R -module. If M has Krull dimension and is σ -compressible, then M is critical.*

Proof. Since M has Krull dimension, M has a critical submodule N (by Lemma 6.1.8(i)). By hypothesis, M is σ -compressible, so there exists an integer n such that $M^{\sigma^n} \hookrightarrow N$. Thus, by Lemma 6.1.8(ii), M^{σ^n} is critical, so by the remark above, M is critical. \square

For the remainder of this section, R will denote a commutative noetherian ring with finite Krull dimension and $S = R[x; \sigma, \delta]$ will be a q -skew polynomial ring. The following result (and indeed Proposition 6.3.6 and Corollary 6.3.7) appears in ‘untwisted’ form in [33] (Lemma 6.9.5, Proposition 6.9.6 and Corollary 6.9.7) for the cases that $S = R[x, \delta]$ or $R[x, x^{-1}; \sigma]$.

Lemma 6.3.3. *Let I be an ideal of R . Then*

$$\left(\frac{S}{IS} \right)^{\sigma^n} \cong \frac{S}{\sigma^{-n}(I)S} \quad \text{for all } n \in \mathbb{Z}.$$

Proof. Recall that $(S/IS)^{\sigma^n}$ is the same abelian group as S/IS , but has multiplication given by $\bar{s} *_{\sigma^n} t = \bar{s} \sigma^n(t)$ for $\bar{s} \in (S/IS)^{\sigma^n}$ and $t \in S$. Consider the map

$$\begin{aligned} \phi : S^{\sigma^n} &\rightarrow \frac{S}{\sigma^{-n}(I)S} \\ s &\mapsto \sigma^{-n}(s) + \sigma^{-n}(I)S \end{aligned}$$

(recall that σ extends to S via $\sigma(x) = q^{-1}x$). Clearly ϕ is an additive map. Let $s^{\sigma^n} \in S^{\sigma^n}$ and $t \in S$. Then

$$\begin{aligned}\phi(s^{\sigma^n} *_{\sigma^n} t) &= \phi(s\sigma^n(t)) = \sigma^{-n}(s\sigma^n(t)) + \sigma^{-n}(I)S \\ &= \sigma^{-n}(s)t + \sigma^{-n}(I)S \\ &= \phi(s^{\sigma^n})t.\end{aligned}$$

Thus ϕ is a module homomorphism. It is clear that ϕ is surjective while $\ker(\phi) = (IS)^{\sigma^n}$. \square

Lemma 6.3.4. *Let M be a right R -module. Then $(M \otimes_R S)^\sigma \cong M^\sigma \otimes_R S$ via*

$$\phi : (M \otimes_R S)^\sigma \longrightarrow M^\sigma \otimes_R S$$

induced by

$$(m \otimes x^n)^\sigma \mapsto m^\sigma \otimes q^n x^n.$$

Proof. The map ϕ is clearly a bijective additive map. To show that ϕ is an S -module homomorphism it is enough to check that

$$\phi((m \otimes x^n)^\sigma *_\sigma x) = \phi((m \otimes x^n)^\sigma) x$$

and

$$\phi((m \otimes x^n)^\sigma *_\sigma r) = \phi((m \otimes x^n)^\sigma) r$$

for $r \in R$. Firstly

$$\begin{aligned}\phi((m \otimes x^n)^\sigma *_\sigma x) &= \phi((m \otimes x^n \sigma(x))^\sigma) \\ &= \phi(m \otimes x^{n+1} q^{-1}) \\ &= m^\sigma \otimes q^{n+1} x^{n+1} q^{-1} \\ &= m^\sigma \otimes q^n x^{n+1}\end{aligned}$$

and

$$\begin{aligned}\phi((m \otimes x^n)^\sigma) x &= (m^\sigma \otimes q^n x^n) x \\ &= m^\sigma \otimes q^n x^{n+1}.\end{aligned}$$

Now let $r \in R$. Then

$$\begin{aligned}
 \phi((m \otimes x^n)^\sigma *_\sigma r) &= \phi((m \otimes x^n \sigma(r))^\sigma) \\
 &= \phi\left(m \otimes \sum_{i=0}^n \binom{n}{i}_q \sigma^{n-i} \delta^i(\sigma(r)) x^{n-i}\right) \\
 &= \phi\left(\sum_{i=0}^n m \binom{n}{i}_q q^i \sigma^{n-i+1} \delta^i(r) \otimes x^{n-i}\right) \\
 &= \sum_{i=0}^n m \binom{n}{i}_q q^i \sigma^{n-i+1} \delta^i(r) \otimes q^{n-i} x^{n-i} \\
 &= \sum_{i=0}^n m \binom{n}{i}_q q^n \sigma^{n-i+1} \delta^i(r) \otimes x^{n-i}.
 \end{aligned}$$

On the other hand

$$\begin{aligned}
 \phi((m \otimes x^n)^\sigma) r &= m^\sigma \otimes q^n x^n r \\
 &= m^\sigma \otimes q^n \sum_{i=0}^n \binom{n}{i}_q \sigma^{n-i} \delta^i(r) x^{n-i} \\
 &= \sum_{i=0}^n m^\sigma *_\sigma \left(q^n \binom{n}{i}_q \sigma^{n-i} \delta^i(r) \right) \otimes x^{n-i} \\
 &= \sum_{i=0}^n m \binom{n}{i}_q q^n \sigma^{n-i+1} \delta^i(r) \otimes x^{n-i}.
 \end{aligned}$$

Therefore ϕ is an S -module homomorphism. \square

Corollary 6.3.5. *Let M be a right R -module. Then*

$$\text{Kdim}(M^{\sigma^n} \otimes_R S) = \text{Kdim}((M \otimes_R S)^{\sigma^n}) = \text{Kdim}(M \otimes_R S)$$

for any $n \in \mathbb{Z}$. \square

Proposition 6.3.6. *Suppose that M is a compressible R -module. Then $M \otimes_R S$ is a σ -compressible right S -module.*

Proof. Note that from Lemma 6.2.5 we have

$$x^n r = \sigma(r) x^n + \text{terms of lower degree},$$

so

$$x^n \sigma^{-n}(r) = r x^n + \text{terms of lower degree}.$$

Let N be a cyclic submodule of M . Then $M \hookrightarrow N$ and since S is a flat left R -module, $M \otimes_R S \hookrightarrow N \otimes_R S$. So, without loss of generality, we may suppose

that M is cyclic. Let $M = R/I$ for some ideal I of R and identify $M \otimes_R S$ with S/IS .

Let L be a nonzero right S -submodule of S/IS and choose $0 \neq y \in L$ such that

$$y = \sum_{j=0}^n m_j \otimes x^j \text{ with } n \text{ minimal.}$$

Then $m_n \neq 0$ and since M is compressible $M \hookrightarrow m_n R$. Therefore, there exists $r \in R$ such that $\text{ann}_R(m_n r) = I$. Consider

$$\begin{aligned} y\sigma^{-n}(r) &= \left(\sum_{j=0}^n m_j \otimes x^j \right) \sigma^{-n}(r) \\ &= m_n \otimes x^n \sigma^{-n}(r) + \text{terms of lower degree} \\ &= m_n r \otimes x^n + \text{terms of lower degree.} \end{aligned}$$

So, by replacing y by $y\sigma^{-n}(r)$, we may assume that $\text{ann}_R(m_n) = I$.

Claim 1: $\text{ann}_R(m_n \otimes x^n) = \sigma^{-n}(I)$

Let $r \in \sigma^{-n}(I)$. Then $r = \sigma^{-n}(s)$ for some $s \in I$ and

$$\begin{aligned} (m_n \otimes x^n)r &= m_n \otimes x^n \sigma^{-n}(s) \\ &= m_n s \otimes x^n + \text{terms of lower degree} \\ &= 0 + \text{terms of lower degree} \\ &= 0, \text{ by the minimality of } n. \end{aligned}$$

Now let $r \in \text{ann}_R(m_n \otimes x^n)$. Then

$$\begin{aligned} (m_n \otimes x^n)r = 0 &\Rightarrow m_n \sigma^n(r) \otimes x^n + \text{terms of lower degree} = 0 \\ &\Rightarrow m_n \sigma^n(r) \otimes x^n = 0 \\ &\Rightarrow m_n \sigma^n(r) = 0. \end{aligned}$$

Therefore $\sigma^n(r) \in \text{ann}_R(m_n) = I$ and $r \in \sigma^{-n}(I)$. Thus $\text{ann}_R(m_n \otimes x^n) = \sigma^{-n}(I)$, as required.

Claim 2: $\text{ann}_R(y) = \text{ann}_R(m_n \otimes x^n)$

Let $r \in \text{ann}_R(m_n \otimes x^n)$. Then $yr \in L$ and yr has degree less than n . Therefore, by the minimality of n , we have $yr = 0$. Now suppose $r \in \text{ann}_R(y)$. Then

$$yr = 0 \Rightarrow r \in \sigma^{-n}(I) = \text{ann}_R(m_n \otimes x^n),$$

by exactly the same argument as used above in Claim 1. Therefore $\text{ann}_R(y) = \text{ann}_R(m_n \otimes x^n)$ as required.

Claim 3: $\text{ann}_S(y) = \sigma^{-n}(I)S$

Certainly $\sigma^{-n}(I)S \subseteq \text{ann}_S(y)$. Suppose that $s \in S \setminus \sigma^{-n}(I)S$ and let

$$s = \sum_{i=0}^k r_i x^i \quad \text{with } r_k \notin \sigma^{-n}(I).$$

Consider the term of highest degree in ys :

$$(m_n \otimes x^n) r_k x^k = m_n \sigma^n(r_k) \otimes x^{n+k} + \dots$$

Since $\sigma^n(r_k) \notin I$ we have that $m_n \sigma^n(r_k) \otimes x^{n+k} \neq 0$. Therefore $ys \neq 0$ and $\text{ann}_S(y) = \sigma^{-n}(I)S$, as required.

Therefore

$$yS \cong \frac{S}{\sigma^{-n}(I)S} \cong \left(\frac{S}{IS} \right)^{\sigma^n}.$$

Thus $(S/IS)^{\sigma^n} \hookrightarrow L$; that is,

$$(M \otimes_R S)^{\sigma^n} \hookrightarrow L,$$

as required. \square

Corollary 6.3.7. *If P is a prime ideal of R , then $(S/PS)_S$ is σ -compressible and critical.*

Proof. The result follows directly from R/P being a compressible R -module ([33] 6.9.3). \square

We will now show that a noetherian module M over a commutative noetherian ring R is critical if and only if it is compressible. This result is presumably well known, but we have been unable to find it in the literature in the exact form that we require. We already have that when M is compressible, M is critical.

Lemma 6.3.8. *Let M be a noetherian module over a commutative noetherian ring R . Then M is cyclic and critical if and only if $M \cong R/P$ for some prime ideal P of R .*

Proof. (\Leftarrow) Suppose that $M \cong R/P$. Then M is certainly cyclic. We will show that M is compressible and therefore critical.

Let $0 \neq B \subseteq M$. Then $B \cong I/P$ for some right ideal I such that $P \subsetneq I \subseteq R$. Choose $c \in I \setminus P$ and let $0 \neq b = c + P \in I/P$. Since $cP \subseteq P$ we have that

$(c + P)P = 0$. Also, if $br = 0$, then $cr \in P$ and since P is prime, $r \in P$. Therefore $\text{ann}_R(b) = P$ and

$$bR \cong R/P \cong M;$$

that is, $M \hookrightarrow B$.

(\Rightarrow) Suppose that M is critical and cyclic, so $M = mR$ for some $m \in M$ and let $\text{ann}_R(m) = I$. We claim that I is prime. Let $0 \neq b, c \in R$ and suppose that $bc \in I$, but $b \notin I$. Consider the map

$$\begin{aligned} M/Mc &\rightarrow Mb \\ n + Mc &\mapsto nb, \end{aligned}$$

which is well defined since $Mcb = mRcb = mbcR = 0$. This map is onto and therefore $\text{Kdim}(Mb) \leq \text{Kdim}(M/Mc)$. Let $\text{Kdim}(M) = \alpha$. Since M is critical, $\text{Kdim}(Mb) = \alpha$ and $\alpha \leq \text{Kdim}(M/Mc)$. However, if $Mc \neq 0$, then $\text{Kdim}(M/Mc) < \alpha$, which is a contradiction, and so $Mc = 0$. Therefore $c \in \text{ann}_R(m) = I$ and $M \cong R/I$, where I is a prime ideal of R . \square

Lemma 6.3.9. *Let M be a critical module over a commutative noetherian ring R . Then there exists a prime ideal P of R such that $\text{ann}_R(m) = P$ for all $m \in M$.*

Proof. Let $0 \neq m, n \in M$. By the proof of Lemma 6.3.8, $\text{ann}_R(m) = P$ and $\text{ann}_R(n) = Q$ for some prime ideals P and Q of R . Suppose that $Q \not\subseteq P$. Then $P + Q \supset P$. Now, M is critical and therefore uniform, so $0 \neq mR \cap nR$. Let $0 \neq c \in mR \cap nR$. Then

$$c(P + Q) = mrP + nsQ \text{ for some } r, s \in R,$$

and since $P = \text{ann}_R(m)$ and $Q = \text{ann}_R(n)$, we have that $c(P + Q) = 0$. Therefore $\text{ann}_R(c) \supset P$ and

$$\text{Kdim}(cR) = \text{Kdim}(R/\text{ann}_R(c)) < \text{Kdim}(R/P) = \text{Kdim}(M).$$

However, $cR \subseteq M$ and M is critical, so $\text{Kdim}(cR) = \text{Kdim}(M)$ and we have a contradiction. Thus, $Q \subseteq P$ and similarly $P \subseteq Q$. \square

Proposition 6.3.10. *Let R be a commutative noetherian ring and let M be a noetherian R -module. Then M is critical if and only if M is compressible.*

Proof. (\Leftarrow) See [33] Lemma 6.9.4.

(\Rightarrow) Suppose $0 \neq N \subsetneq M$. Since M is finitely generated, $M = m_1R + m_2R + \dots + m_nR$ for some $m_i \in M$. By Lemma 6.3.9, $\text{ann}_R(m_j) = P$ for some prime ideal P of R . Consider $I_j := \{r \in R \mid m_jr \in N\}$. Then since P annihilates m_j , we have $P \subseteq I_j$. Then

$$\frac{R}{I_j} \cong \frac{m_jR + N}{N}$$

and since $m_jR + N$ is critical (being a submodule of the critical module M), we have

$$\text{Kdim} \left(\frac{R}{I_j} \right) = \text{Kdim} \left(\frac{m_jR + N}{N} \right) < \text{Kdim}(M).$$

If $I_j = P$, then $\text{Kdim}(R/I_j) = \text{Kdim}(m_jR) = \text{Kdim}(M)$, which is a contradiction. Therefore $I_j \supsetneq P$ and so $I := I_1 \cap I_2 \cap \dots \cap I_n \supset P$. Let $c \in I \setminus P$. Then

$$\begin{aligned} Mc &= m_1Rc + m_2Rc + \dots + m_nRc \\ &= m_1cR + m_2cR + \dots + m_ncR \\ &\subseteq N. \end{aligned}$$

Also, $M \cong Mc$ via $m \mapsto mc$. Thus $M \cong Mc \subseteq N$ and therefore $M \cong Mc \hookrightarrow N$, as required. \square

Corollary 6.3.11. *Let M be a noetherian R -module. If M is critical, then the right S -module $(M \otimes_R S)_S$ is critical.*

Proof. Since M is critical, M is compressible by Proposition 6.3.10. So by Proposition 6.3.6, $(M \otimes_R S)_S$ is σ -compressible and is therefore critical by Lemma 6.3.2. \square

Definition 6.3.12. Let R be a commutative noetherian ring and let S be a q -skew polynomial ring over R ; that is, $S = R[x; \sigma, \delta]$. Let M be an arbitrary R -module and consider an element $0 \neq m \in M \otimes_R S$. Then m can be written uniquely in the form

$$m = \sum_{i=0}^n m_i \otimes x^i \quad \text{with } m_n \neq 0.$$

We say that the **degree** of m , $\deg(m)$, equals n and that m_n is the **leading coefficient** of m . If $m = 0$, we define $\deg(m) := -1$ and say that m has leading coefficient 0. Let I be a submodule of $M \otimes_R S$ and define:

$$I_n := \{m \in I = M \otimes_R S \mid \deg(m) \leq n\};$$

$$\lambda_n(I) := \{m_n \in M \mid m_n \text{ is the leading coefficient of some } m \in I_n\};$$

$$\lambda(I) := \cup \lambda_n(I).$$

Then $\lambda_n(I)$ is an R -submodule of M , called the n th **leading submodule** of I .

Let M be a module over a commutative noetherian ring R . Before we can express the Krull dimension of $M \otimes_R S$ in terms of the Krull dimension of modules of the form $M' \otimes_R S$ where M' is a critical subfactor of M with Krull dimension less than that of M , we should consider what such a subfactor could be. The following two results mirror results in [16] Section I and the proofs here are very similar to those in [16].

Proposition 6.3.13. *Let M be a noetherian R -module and let I and J be S -submodules of $M \otimes_R S$ such that $I \subseteq J$. Suppose that $0 \neq N$ is a noetherian R -module such that $N \otimes_R S$ is isomorphic to an S -module subfactor C of J/I . Then there exists a nonzero subfactor C of $\lambda(J)/\lambda(I)$ and an integer α such that $C^{\sigma^\alpha} \subseteq N$.*

Proof. (cf. [16] Proposition 1.4.) We may assume that $J/I \cong N \otimes_R S$ (if this is not the case, we may enlarge I or reduce J). Consider the following chain of R -submodules of $N \otimes S$:

$$N \otimes R \subseteq N \otimes R + N \otimes Rx \subseteq N \otimes R + N \otimes Rx + N \otimes Rx^2 \subseteq \dots$$

Let $\beta_i = \sum_{j=0}^i N \otimes Rx^j$ for $i \geq 0$. Then $N \otimes_R S = \cup \beta_i$ and we have a surjective additive map

$$\begin{aligned} \psi : N^{\sigma^{i+1}} &\longrightarrow \beta_{i+1}/\beta_i \\ n &\mapsto n \otimes x^{i+1} + \beta_i. \end{aligned}$$

Let $r \in R$ and $n \in N^{\sigma^{i+1}}$. Then

$$\begin{aligned} \psi(n)r &= (n \otimes x^{i+1} + \beta_i)r \\ &= n\sigma^{i+1}(r) \otimes x^{i+1} + \sum_{j=1}^{i+1} b \otimes \binom{i+1}{j}_q \sigma^{i+1-j} \delta^j(r) x^{i+1-j} + \beta_i \\ &= n\sigma^{i+1}(r) \otimes x^{i+1} + \beta_i \\ &= \psi(n\sigma^{i+1}(r)) \\ &= \psi(n *_{\sigma^{i+1}}(r)) \end{aligned}$$

and ψ is a module homomorphism. Also,

$$\begin{aligned}\ker(\psi) &= \{n \in N \mid n \otimes x^{i+1} + \beta_i = 0\} \\ &= \{n \in N \mid n \otimes x^{i+1} = 0\} \\ &= 0.\end{aligned}$$

Therefore, $\beta_{i+1}/\beta_i \cong N^{\sigma^{i+1}}$ for $i \geq 0$ and $\beta_0 \cong N$. Therefore, in $M \otimes_R S$, there exist R -submodules

$$I = N_0 \subsetneq N_1 \subsetneq \dots \subsetneq J$$

such that $\cup N_i = J$ and $N_{i+1}/N_i \cong N^{\sigma^{i+1}}$ for $i \geq 0$.

From Definition 6.3.12, we have an ascending chain of R -submodules of $M \otimes_R S$:

$$I_0 \subseteq I_1 \subseteq I_2 \subseteq \dots$$

and

$$\lambda_0(I) \subseteq \lambda_1(I) \subseteq \lambda_2(I) \subseteq \dots$$

is an ascending chain of submodules of M . Similarly we have an ascending chain J_j of R -submodules of $M \otimes_R S$, and an ascending chain $\lambda_j(J)$ of submodules of M . Since M is noetherian there exists an integer p such that

$$\lambda_n(I) = \lambda_p(I) \text{ and } \lambda_n(J) = \lambda_p(J) \text{ for all } n \geq p$$

and

$$\lambda_n(I) = \lambda(I) \text{ and } \lambda_n(J) = \lambda(J) \text{ for all } n \geq p.$$

Now consider the map

$$\begin{aligned}\theta &: J_n \longrightarrow \left[\frac{\lambda_n(J)}{\lambda_n(I)} \right]^{\sigma^n} \\ f = \sum_{i=0}^n f_i \otimes x^i &\mapsto f_n + \lambda_n(I)^{\sigma^n}.\end{aligned}$$

We claim that this map is an R -module homomorphism. It is clearly a surjective additive map. Let $r \in R$. Then

$$\begin{aligned}\theta(fr) &= \theta \left(\sum_{i=0}^n f_i \otimes x^i r \right) \\ &= \theta(f_n \sigma^n(r) \otimes x^n + \text{terms of lower degree}) \\ &= f_n \sigma^n(r) + \lambda_n(I)^{\sigma^n} \\ &= \left[f_n + \lambda_n(I)^{\sigma^n} \right] *_{\sigma^n} r \\ &= \theta(f) *_{\sigma^n} r.\end{aligned}$$

We claim that $\ker(\theta) = I_n + J_{n-1}$. Clearly $I_n + J_{n-1} \subseteq \ker(\theta)$. Suppose $f = \sum f_i \otimes x^i \in \ker(\theta) = \{f \in J_n \mid \theta(f) = 0\}$. Then $f_n \in \lambda_n(I)$; that is, f_n is the leading coefficient of some $g \in I_n$. Let $g = f_n \otimes x^m + \sum_{i=0}^{m-1} g_i \otimes x^i$ for some $m \leq n$. Then

$$gx^{n-m} = f_n \otimes x^n + \sum_{i=0}^{m-1} g_i \otimes x^{i+n-m} \in I_n$$

and

$$f - gx^{n-m} = \sum_{i=0}^{n-1} f_i \otimes x^i - \sum_{i=0}^{m-1} g_i \otimes x^{i+n-m} \in I_{n-1} \subseteq J_{n-1}.$$

Also, $gx^{n-m} \in I_n$. Therefore $f \in I_n + J_{n-1}$ and $\ker\theta = I_n + J_{n-1}$. Thus

$$\frac{J_n}{I_n + J_{n-1}} \cong \left[\frac{\lambda_n(J)}{\lambda_n(I)} \right]^{\sigma^n}.$$

We also have the isomorphisms

$$\frac{I + J_n}{I + J_{n-1}} \cong \frac{J_n}{J_n \cap (I + J_{n-1})} \cong \frac{J_n}{I_n + J_{n-1}}.$$

So for $n \geq p$,

$$\frac{I + J_n}{I + J_{n-1}} \cong \frac{J_n}{I_n + J_{n-1}} \cong \left[\frac{\lambda(J)}{\lambda(I)} \right]^{\sigma^n}.$$

The R -module $\{f \in M \otimes_R S \mid \deg(f) \leq p\}$ is noetherian and thus J_p is noetherian, as is $(I + J_p)/I$. Therefore, the following chain of submodules in $(I + J_p)/I$ must terminate:

$$\frac{N_0}{I} \cap \left(\frac{I + J_p}{I} \right) \subseteq \frac{N_1}{I} \cap \left(\frac{I + J_p}{I} \right) \subseteq \frac{N_2}{I} \cap \left(\frac{I + J_p}{I} \right) \subseteq \dots$$

So

$$\frac{N_{q+1}}{I} \cap \left(\frac{I + J_p}{I} \right) = \frac{N_q}{I} \cap \left(\frac{I + J_p}{I} \right)$$

for some positive integer q , so that $N_{q+1} \cap J_p \subseteq N_q$. Now consider the following R -submodules of $M \otimes_R S$:

$$A := N_{q+1} + J_p \text{ and } B := N_q + J_p.$$

Then $B \subsetneq A$ and

$$\frac{A}{B} = \frac{N_{q+1} + J_p}{N_q + J_p} \cong \frac{N_{q+1}}{N_{q+1} \cap (N_q + J_p)} = \frac{N_{q+1}}{N_q}$$

where the last equality here is due to the modular law. Therefore,

$$\frac{A}{B} \cong N^{\sigma^{q+1}}.$$

Now, the R -module N_{q+1}/I is noetherian and J_p is finitely generated. Therefore A/I is a finitely generated R -submodule of J/I and $A \subseteq I + J_n$ for some $n \geq p$. Thus we obtain two chains of R -modules:

$$I + J_p \subseteq B \subsetneq A \subseteq I + J_n$$

and

$$I + J_p \subseteq I + J_{p+1} \subseteq \dots \subseteq I + J_n.$$

The Schreier refinement theorem enables us to find refinements of the two chains which are equivalent. Thus, some submodule F of A/B is isomorphic to a subfactor G of one of the modules $(I + J_{m+1}) / (I + J_m)$ for some $p \leq m \leq n$. We have that

$$\frac{I + J_{m+1}}{I + J_m} \cong \left[\frac{\lambda(J)}{\lambda(I)} \right]^{\sigma^m}.$$

So there exists a subfactor C of $\lambda(J)/\lambda(I)$ such that $C^{\sigma^m} \cong G$. Also, since $A/B \cong N^{\sigma^{q+1}}$, there is a submodule E of N such that $E^{\sigma^{q+1}} \cong F$. Therefore

$$C^{\sigma^m} \cong G \cong F \cong E^{\sigma^{q+1}}$$

and

$$C^{\sigma^{m-(q+1)}} \cong E \subseteq N.$$

□

Proposition 6.3.14. *Let M be an α -critical R -module for some ordinal α , and let*

$$M \otimes_R S \supseteq N_1 \supseteq N_2 \supseteq \dots \supseteq N \supsetneq 0$$

be a chain of R -submodules of $M \otimes_R S$ with N an S -submodule of $M \otimes_R S$. Then there exists a positive integer p such that for any integer $j \geq p$, all finitely generated R -module subfactors of N_j/N_{j+1} have Krull dimension less than α .

Proof. (cf. [16] Proposition 1.6.) Set $M_n := \{f \in M \otimes_R S \mid \deg(f) \leq n\}$ for each $n = 0, 1, 2, \dots$. Then $M_{n+1}/M_n \cong M^{\sigma^{n+1}}$ via the map

$$\begin{aligned} \psi &: M_{n+1} \longrightarrow M^{\sigma^{n+1}} \\ \sum_{i=0}^{n+1} m_i \otimes x^i &\mapsto m_{n+1}. \end{aligned}$$

Since M is α -critical, $M^{\sigma^{n+1}}$ is α -critical and M_{n+1}/M_n is α -critical.

Having noted the above, the reader is referred to [16] Proposition 1.6 for the completion of this proof. \square

At this point in the work of [16] (and [15]), a certain subset of the critical modules over the ring is considered. Given an arbitrary ring R and a q -skew polynomial ring S , an R -module M is said to be **S -clean** if it is critical as an R -module and $M \otimes_R S$ is critical as an S -module. This classification is necessary in the set-ups considered in [16] and [15], as there is no guarantee that starting with a critical module M over the ring will yield a critical module when tensored with the skew polynomial ring being considered. However, in the case that R is commutative and S is a q -skew polynomial ring, it is clear from Corollary 6.3.11 that the clean modules are precisely the critical modules.

Recall that a **minor subfactor** of an R -module M is any submodule of a proper factor of M .

Definition 6.3.15. Let R be a commutative noetherian ring and M, N be critical R -modules. Define $h(M, N) := 1$ if M is isomorphic to a minor subfactor of N but no nonzero submodule of M is isomorphic to a minor subfactor of a critical minor subfactor of N .

In the case that $h(M, N) = 1$ does not hold, we say that $h(M, N) \neq 1$.

Lemma 6.3.16. *Let R be a commutative noetherian ring and M be a critical noetherian R -module. Let*

$$M \supseteq N_1 \supseteq N_2 \supseteq \dots \supseteq N \supsetneq 0$$

be a chain of R -submodules of M . Then there exists a positive integer p such that for all integers $j \geq p$, there are no critical subfactors X of N_j/N_{j+1} satisfying $h(X, M) = 1$.

Proof. This is a special case of [15] Lemma 4.3. \square

Lemma 6.3.17. *Let N be a noetherian R -module and let D be a nonzero S -subfactor of $N \otimes_R S$. Then there exists a nonzero subfactor C of N and an integer p such that C^{σ^p} is isomorphic to a submodule of D_R .*

Proof. Set $N_i := \{f \in N \otimes_R S \mid \deg(f) \leq i - 2\}$ for all positive integers i (then $N_1 = 0$) and write $D = E/F$ for some S -submodules $F \subsetneq E$ in $N \otimes_R S$. Then

we have two chains of R -submodules:

$$0 \subseteq F \subsetneq E \subseteq N \otimes_R S$$

and

$$0 = N_1 \subseteq N_2 \subseteq N_3 \subseteq \dots$$

We use [15] Proposition 3.2(a) to obtain a common refinement of these chains. So there exist chains of R -submodules

$$\begin{aligned} 0 &= V_{01} \leq V_{02} \leq \dots \leq F; \\ F &= V_{11} \leq V_{12} \leq \dots \leq E; \\ E &= V_{21} \leq V_{22} \leq \dots \leq N \otimes_R S \end{aligned}$$

and

$$N_j = W_{0j} \leq W_{1j} \leq W_{2j} \leq W_{3j} = N_{j+1},$$

such that

$$\frac{V_{i,j+1}}{V_{ij}} \cong \frac{W_{i+1,j}}{W_{ij}}$$

for all i, j . Also, since $\cup N_j = N \otimes_R S$, we have

$$\cup V_{0j} = F, \quad \cup V_{1j} = E \quad \text{and} \quad \cup V_{2j} = N \otimes_R S.$$

Let k be the least positive integer such that $V_{1k} \supsetneq F$ (so $k \geq 2$) and set $C' = V_{1k}/F$. Then $0 \neq C'$ is a R -submodule of D and

$$C' = \frac{V_{1k}}{F} = \frac{V_{1k}}{V_{1,k-1}} \cong \frac{W_{2,k-1}}{W_{1,k-1}}.$$

Thus, C' is isomorphic to a subfactor of N_k/N_{k-1} , which (as seen in the proof of Proposition 6.3.14) is isomorphic to N^{σ^k} . So there exists a nonzero subfactor C of N such that $C^{\sigma^k} \cong C' \subseteq D$. \square

Corollary 6.3.18. *Let N be a noetherian R -module and $0 \neq D$ be a S -module subfactor of $N \otimes_R S$. Then there exists a critical subfactor M of N , a critical R -submodule $E \subseteq D$ and an integer p such that $M^{\sigma^p} \cong E$.*

Proof. By Lemma 6.3.17, there exists a nonzero subfactor C of N and an integer p such that $C^{\sigma^p} \hookrightarrow D_R$. Now, C is noetherian and so contains a critical submodule M and $M^{\sigma^p} \subseteq C^{\sigma^p} \hookrightarrow D_R$. Therefore, there exists a critical submodule E of D such that $M^{\sigma^p} \cong E$. \square

Proposition 6.3.19. *Let N be a critical noetherian R -module with finite Krull dimension and let \mathcal{N} be the family of critical minor subfactors of N . If N is not simple, then*

$$\text{Kdim}(N \otimes_R S) = \max\{\text{Kdim}(M \otimes_R S) \mid M \in \mathcal{N}\} + 1.$$

Proof. First note that \mathcal{N} is non-empty. Let $\beta = \text{Kdim}(N)$. Then β is finite and therefore $\text{Kdim}(N \otimes_R S)$ is finite. Let

$$\alpha = \max\{\text{Kdim}(M \otimes_R S) \mid M \in \mathcal{N}\}$$

and choose $M \in \mathcal{N}$ such that $\text{Kdim}(M \otimes_R S) = \alpha$. Since $N \otimes_R S$ is critical and $M \otimes_R S$ is isomorphic to a minor subfactor of $N \otimes_R S$, we have

$$\text{Kdim}(N \otimes_R S) > \text{Kdim}(M \otimes_R S) = \alpha.$$

Thus $\text{Kdim}(N \otimes_R S) \geq \alpha + 1$.

We now suppose that $\text{Kdim}(N \otimes_R S) > \alpha + 1$ and aim to obtain a contradiction. Since $\text{Kdim}(N \otimes_R S) > \alpha + 1$, there exists an S -submodule C of $N \otimes_R S$ such that

$$\text{Kdim}\left(\frac{N \otimes_R S}{C}\right) > \alpha.$$

Then there exists a chain of S -submodules

$$N \otimes_R S \supseteq C_1 \supseteq C_2 \supseteq \dots \supseteq C > 0$$

such that $\text{Kdim}(C_j/C_{j+1}) \geq \alpha$ for infinitely many j . By refining this chain, we may assume that each C_j/C_{j+1} is critical. Now, by Proposition 6.3.14, there exists a positive integer p such that for any $j > p$, all finitely generated R -subfactors of C_j/C_{j+1} have Krull dimension less than β .

Sitting inside N we have a chain of R -submodules

$$N \supseteq \lambda(C_1) \supseteq \lambda(C_2) \supseteq \dots \supseteq \lambda(C) > 0.$$

Lemma 6.3.16 says that there is a positive integer q such that for all integers $j \geq q$, there are no critical subfactors X of $\lambda(C_j)/\lambda(C_{j+1})$ satisfying $h(X, N) = 1$. Choose $m \geq \max\{p, q\}$ such that $\text{Kdim}(C_m/C_{m+1}) \geq \alpha$. By Corollary 6.3.18, there is a critical subfactor M of N , a critical R -submodule M' of C_m/C_{m+1} and an integer k such that $M^{\sigma^k} \cong M'$. Since $m \geq p$, we have $\text{Kdim}(M') < \beta$ and $\text{Kdim}(M) = \text{Kdim}(M^{\sigma^k}) = \text{Kdim}(M') < \beta = \text{Kdim}(N)$. Now, M is

a subfactor of the critical module N , so M must be a minor subfactor of N . Therefore $M \in \mathcal{N}$ and

$$\text{Kdim}(M \otimes_R S) \leq \alpha.$$

Now, by Corollary 6.3.5, $\text{Kdim}(M \otimes_R S) = \text{Kdim}(M^{\sigma^k} \otimes_R S)$ and since $M^{\sigma^k} \cong M'$, we have that $\text{Kdim}(M' \otimes_R S) = \text{Kdim}(M \otimes_R S) \leq \alpha$.

Now let $0 \neq E = M'S$. Then E is an S -submodule of C_m/C_{m+1} and

$$\text{Kdim}(E) = \text{Kdim}(C_m/C_{m+1}) \geq \alpha.$$

The S -module E is a homomorphic image of the critical module $M' \otimes_R S$, but $\text{Kdim}(E) \geq \text{Kdim}(M' \otimes_R S)$, so it cannot be a proper homomorphic image; that is, $E \cong M' \otimes_R S$ and $\text{Kdim}(E) = \text{Kdim}(M' \otimes_R S) = \alpha$.

Since $M' \otimes_R S$ is isomorphic to a submodule of C_m/C_{m+1} , there exists a nonzero subfactor X of $\lambda(C_m)/\lambda(C_{m+1})$ and an integer l such that $X^{\sigma^l} \cong M'$ (Proposition 6.3.13). Then X is a minor subfactor of N and since X certainly contains a critical module, there is no loss of generality in assuming X to be critical, forcing $h(X, N) \neq 1$, since $m \geq q$. Thus, there exists a nonzero submodule $Y \subseteq X$ and a critical minor subfactor G of N such that Y is isomorphic to a minor subfactor of G . Then $G \in \mathcal{N}$ and $Y \otimes_R S$ is isomorphic to a minor subfactor of the critical module $G \otimes_R S$. Therefore

$$\text{Kdim}(Y \otimes_R S) < \text{Kdim}(G \otimes_R S) \leq \alpha.$$

Also, $Y \leq X$ and $X^{\sigma^l} \leq M'$, so there is a submodule Y' of M' such that $Y^{\sigma^l} \cong Y'$. So, by Corollary 6.3.5, $\text{Kdim}(Y \otimes_R S) = \text{Kdim}(Y^{\sigma^k} \otimes_R S) = \text{Kdim}(Y' \otimes_R S)$. Now, $Y' \otimes_R S$ is isomorphic to a nonzero submodule of the critical module $M' \otimes_R S$. Therefore

$$\text{Kdim}(Y \otimes_R S) = \text{Kdim}(Y' \otimes_R S) = \text{Kdim}(M' \otimes_R S) = \alpha,$$

which is a contradiction.

Therefore $\text{Kdim}(B \otimes_R S) = \alpha + 1$. □

Proposition 6.3.19 gives an expression for the Krull dimension of $N \otimes_R S$ in terms of the Krull dimensions of S -modules of the form $M \otimes_R S$, where M is a critical minor subfactor of N . We want to use this expression in an inductive process to obtain the Krull dimension of $N \otimes_R S$ in terms of the Krull dimensions of modules of the form $M' \otimes_R S$, where M' is a simple subfactor of N . In [16], the notion of the height of a simple module is defined to keep track of the number of steps in the inductive process. We make the corresponding definition here.

Definition 6.3.20. Let R be a commutative noetherian ring and $S = R[x; \sigma, \delta]$ be a q -skew polynomial ring. Let M be a simple R -module and N be an arbitrary R -module. We define $h_S(M : N)$ to be the supremum of those nonnegative integers n for which there exists a sequence

$$M = M_0, M_1, M_2, \dots, M_n$$

of critical R -modules such that M_i is isomorphic to a minor subfactor of M_{i+1} for $i = 0, 1, \dots, n-1$ while M_n is isomorphic to a subfactor of N . Such a sequence of R -modules is called a **critical sequence** of M with respect to N .

When $N = R$ we will see in Corollary 6.3.25 below that $h_S(M : R)$ is the usual height of a prime ideal P , where $M \cong R/P$.

Theorem 6.3.21. *Let N be a nonzero noetherian R -module with finite Krull dimension. Then*

$$\text{Kdim}(N \otimes_R S) = \max\{\text{Kdim}(M \otimes_R S) + h_S(M : N) \mid M \in \mathcal{M}\}$$

where \mathcal{M} is the set of simple subfactors of N .

Proof. This proof follows exactly that of [16] Theorem 3.1 and there is nothing to be gained by repeating it here. \square

Corollary 6.3.22. *Let R be a commutative noetherian ring and let $S = R[x; \sigma, \delta]$ be a q -skew polynomial ring. Then*

$$\text{Kdim}(S) = \max\{\text{Kdim}(M \otimes_R S) + h_S(M : R) \mid M \in \mathcal{M}\}$$

where \mathcal{M} is the family of simple R -modules. \square

Let P be a prime ideal in a commutative noetherian ring R and recall that the height of P , written $\text{ht}(P)$, is the supremum of the lengths of all finite chains $P = P_0 \supsetneq P_1 \supsetneq \dots \supsetneq P_n$ of prime ideals of R . We introduce a notion of height for a simple R -module M .

Definition 6.3.23. Let M be a simple R -module such that $M \cong R/P$ for some maximal ideal P of R . Let N be an arbitrary R -module and define $h(M : N)$ to be the supremum of the lengths of all finite chains $P = P_0 \supseteq P_1 \supseteq \dots \supseteq P_n$ of prime ideals of R such that R/P_n is isomorphic to a subfactor of N .

Note that $h(M : R) = \text{ht}(P)$.

Proposition 6.3.24. *Let R be a commutative noetherian ring and $S = R[x; \sigma, \delta]$ be a q -skew polynomial ring. Let M be a simple R -module and let N be an arbitrary R -module. Then*

$$h_S(M : N) = h(M : N).$$

Proof. Let P be the maximal ideal of R such that $M \cong R/P$. Consider a chain of prime ideals in R :

$$P = P_0 \supsetneq P_1 \supsetneq \dots \supsetneq P_n$$

where R/P_n is isomorphic to a subfactor of N . Then we have a chain of critical R -modules

$$M \cong R/P, R/P_1, \dots, R/P_n$$

with R/P_n isomorphic to a subfactor of N . Furthermore,

$$\frac{R}{P_i} \cong \frac{R/P_{i+1}}{P_i/P_{i+1}} \text{ for } i = 0, \dots, n-1$$

and R/P_i is isomorphic to a minor subfactor of R/P_{i+1} for $i = 0, \dots, n-1$. So we have constructed a critical sequence of M with respect to N . Therefore, $h(M : N) \leq h_S(M : N)$.

Now consider a critical sequence for M ; that is, a sequence of critical R -modules $M = M_0, M_1, \dots, M_n$ such that M_i is isomorphic to a minor subfactor of M_{i+1} for $i = 0, \dots, n-1$ and M_n is isomorphic to a subfactor of N . Consider M_0 . By Lemma 6.3.9 there exists a prime ideal P_0 of R such that $\text{ann}_R(a_0) = P_0$ for all $0 \neq a_0 \in M_0$. In fact $P = P_0$ and $M_0 \cong R/P_0$. Now, M_0 is isomorphic to a minor subfactor of M_1 ; therefore $M = M_0 \cong F/E$ for some $E \subsetneq F \subseteq M_1$. Again by Lemma 6.3.9, there exists a prime ideal P_1 such that $\text{ann}_R(a_1) = P_1$ for all $a_1 \in M_1$. Since P_1 annihilates everything in M_1 , we have that P_1 annihilates everything in F and therefore annihilates everything in M_0 . Thus $P_0 \supset P_1$.

Continuing in this manner we obtain a descending chain of prime ideals

$$P = P_0 \supsetneq P_1 \supsetneq \dots \supsetneq P_n,$$

where P_n is the prime ideal which annihilates everything in M_n . Therefore R/P_n sits inside M_n which is isomorphic to a subfactor of N , so R/P_n is isomorphic to a subfactor of N . Therefore

$$h(M : N) = h_S(M : N).$$

□

In particular, we have the following Corollary.

Corollary 6.3.25. *Let R be a commutative noetherian ring and M be a simple R -module such that $M \cong R/P$ for some prime ideal P . Then*

$$h_S(M : R) = \text{ht}(P).$$

Corollary 6.3.26. *Let R be a commutative noetherian ring and $S = R[x; \sigma, \delta]$ be a q -skew polynomial ring. Then*

$$\text{Kdim}(S) = \max\{\text{Kdim}((R/P) \otimes_R S) + \text{ht}(P) \mid P \text{ is a maximal ideal of } R\}.$$

□

We now seek a condition on the maximal ideals of R which will determine the Krull dimension of a q -skew polynomial ring over R .

Lemma 6.3.27. *Let M be a noetherian R -module. Then*

$$\text{Kdim}(M \otimes_R S) \leq \text{Kdim}(M) + 1.$$

Proof. Let $S_a(\mathcal{L}(M))$ be the set of eventually constant ascending chains of submodules of M (\mathcal{L} being the lattice of submodules of M). This is a poset via the relation $\{A_i\} \geq \{B_i\}$ if and only if $A_i \geq B_i$ for all i , for $\{A_i\}, \{B_i\} \in S_a(\mathcal{L}(M))$.

Let N be a submodule of $M \otimes_R S$ and consider the ascending chain $\{\lambda_i(N)\}$ of submodules of M . Since M is noetherian, $\{\lambda_i(N)\}$ belongs to $S_a(\mathcal{L}(M))$. Thus we have a map $\mathcal{L}(M \otimes_R S) \rightarrow S_a(\mathcal{L}(M))$, which we claim preserves proper inclusion.

Let $A, B \in \mathcal{L}(M \otimes_R S)$ with $A \subseteq B$ and $A_n = B_n$ for all $n \geq 0$. Suppose $A \neq B$. Choose $y \in B \setminus A$ of least possible degree p , say. Then

$$y = \sum_{i=0}^p b_i \otimes x^i \text{ with } 0 \neq b_p \in B_p = A_p.$$

Then there exists $g \in A$ such that

$$g = b_p \otimes x^p + \sum_{i=0}^{p-1} a_i \otimes x^i.$$

Then $0 \neq y - g \in B \setminus A$ and has smaller degree than y , which contradicts the minimality of p . Therefore the map

$$\mathcal{L}(M \otimes_R S) \rightarrow S_a(\mathcal{L}(M))$$

preserves proper inclusion.

We may now use [33] 6.1.16 and 6.1.17 to obtain the required result; that is,

$$\text{Kdim}(M \otimes_R S) \leq \text{Kdim}(M) + 1.$$

□

Corollary 6.3.28. *Let R be a commutative noetherian ring and $S = R[x; \sigma, \delta]$ be a q -skew polynomial ring. If M is a simple R -module, then*

$$0 \leq \text{Kdim}(M \otimes_R S) \leq 1.$$

In fact, $M \otimes_R S$ is either 0-critical or 1-critical; that is, $M \otimes_R S$ is either simple or 1-critical.

Proof. This result follows from Lemma 6.3.27 and Corollary 6.3.11. □

The following definition is given by Woodward in [38].

Definition 6.3.29. Let R be a commutative noetherian ring and $S = R[x; \sigma, \delta]$ be a q -skew polynomial ring. Let P be a prime ideal of R . Then P is **quasistable** if there exists a right ideal I of S such that $I \cap R = P$ and I strictly contains PS .

Though the above definition of quasistable is useful for our purposes, in a more ‘hands on’ situation it is not always easy to check. Theorem 6.3.31 ([38] Theorem 4.2) gives two equivalent conditions to P being quasistable which may be more accessible. First we require the following definition.

Definition 6.3.30. The q -characteristic of a ring R is the least $t \in \mathbb{N}$ such that $\binom{t}{1}_q = q^{t-1} + q^{t-2} + \dots + 1 = 0$. If there is no such natural number, we say that the q -characteristic of R is 0. Note that the 1-characteristic of R is just the usual characteristic of the ring.

Let P be a prime ideal of R . Then P has associated to it a set $n(P)$ of natural numbers dependent on the q -characteristic t and the 1-characteristic p . Define

$$n(P) := \begin{cases} \{1\} & \text{if } t = 0, \\ \{1, t\} & \text{if } t > 0 \text{ and } p = 0, \\ \{1, t, tp, tp^2, \dots\} & \text{if } t > 0 \text{ and } p > 0. \end{cases}$$

Theorem 6.3.31. *Let R be a commutative noetherian ring, $S = R[x; \sigma, \delta]$ be a q -skew polynomial ring and P be a prime ideal of R . The following are equivalent:*

(i) P is quasistable;

(ii) there exist finitely many integers $n_0, \dots, n_l \in n(P)$ and $a_0, \dots, a_l \in R \setminus P$ and $b \in R$ such that P is stable under $\left(\sum_{j=0}^l a_j \delta^{n_j} + b\right) \sigma^{-n_0}$ and under $\sigma^{n_j - n_0}$ for all j ;

(iii) either P is not stable under σ or if P is σ -stable, there exist finitely many integers $n_0, \dots, n_l \in n(P)$ and $a_0, \dots, a_l \in R \setminus P$ such that P is stable under $\sum_{j=0}^l a_j \delta^{n_j}$.

Proof. See [38] Theorem 4.2. □

In [38], Woodward is concerned with calculating the global dimension of q -skew polynomial rings over a commutative noetherian ring. The main result in [38], Theorem 5.5, establishes a criterion on the maximal ideals of the original ring which determines the global dimension of the q -skew polynomial ring.

Theorem 6.3.32. *Let R be a commutative noetherian ring with $\text{gldim}(R) = n < \infty$ and $S = R[x; \sigma, \delta]$ be a q -skew polynomial ring. Then $\text{gldim}(S) = n + 1$ if and only if R possesses a quasistable maximal ideal of height n .* □

We claim that the same criterion on the maximal ideals of R will establish the Krull dimension of a q -skew polynomial ring over R .

Lemma 6.3.33. *Let P be a maximal ideal of R . Then $(R/P) \otimes_R S$ is 1-critical if and only if P is quasistable.*

Proof. Note that $(R/P) \otimes_R S \cong S/PS$.

(\Rightarrow) Suppose $(R/P) \otimes_R S$ is 1-critical. Then S/PS is not simple and there exists a right ideal I of S such that $PS \subsetneq I \subsetneq S$. Consider $I \cap R$. Certainly $P \subseteq I \cap R$ and since P is maximal we have either $P = I \cap R$ or $R = I \cap R$. However, since $I \neq S$, we have $I \cap R \neq R$ and therefore $I = P \cap R$; that is, P is quasistable.

(\Leftarrow) Suppose P is quasistable. Then there exists a right ideal I of S such that $I \cap R = P$ and I strictly contains PS . If $I = S$ then $I \cap R = R$, which is a contradiction. Therefore $PS \subsetneq I \subsetneq S$ and S/PS is not simple. Thus by Corollary 6.3.28, S/PS is 1-critical. □

Theorem 6.3.34. *Let R be a commutative noetherian ring with $\text{Kdim}(R) = n < \infty$ and let $S = R[x; \sigma, \delta]$ be a q -skew polynomial ring. Then $\text{Kdim}(S) = n + 1$ if and only if R possesses a quasistable maximal ideal of height n .*

Proof. (\Rightarrow) Suppose $\text{Kdim}(S) = \text{Kdim}(R) + 1$. Then there exists a maximal ideal P of R such that

$$\text{Kdim}((R/P) \otimes_R S) + \text{ht}(P) = \text{Kdim}(R) + 1.$$

Since $\text{ht}(P) \leq \text{Kdim}(R)$ we have $\text{Kdim}((R/P) \otimes_R S) \geq 1$. So by Corollary 6.3.28, $\text{Kdim}((R/P) \otimes_R S) = 1$ and so $\text{ht}(P) = n$. Thus, by Lemma 6.3.33, P is a quasistable maximal ideal of height n .

(\Leftarrow) Suppose P is a quasistable maximal ideal of height n in R .

Then $\text{Kdim}((R/P) \otimes_R S) = 1$ and

$$\text{Kdim}((R/Q) \otimes_R S) \leq \text{Kdim}((R/P) \otimes_R S) \quad \text{and} \quad \text{ht}(Q) \leq \text{ht}(P)$$

for all maximal ideals Q of R . Thus

$$\begin{aligned} \text{Kdim}(S) &= \max\{\text{Kdim}((R/Q) \otimes_R S) + \text{ht}(Q) \mid Q \text{ is a maximal ideal of } R\} \\ &= \text{Kdim}((R/P) \otimes_R S) + \text{ht}(P) \\ &= n + 1. \end{aligned}$$

□

Recall the following result.

Proposition 6.3.35. *Let R be a commutative noetherian ring with finite global dimension $\text{gldim}(R)$ equal to n . Then $\text{Kdim}(R) = n$.* □

Thus we obtain the following corollary.

Corollary 6.3.36. *Let R be a commutative noetherian ring with $\text{gldim}(R) = \text{Kdim}(R) = n < \infty$. Let $S = R[x; \sigma, \delta]$ be a q -skew polynomial ring with σ an automorphism of R . Then*

$$\text{Kdim}(S) = \text{gldim}(S).$$

Proof. The result follows from [38] Theorem 5.5 and Theorem 6.3.34 above. □

We conclude with two examples of q -skew polynomial rings, one of which has Krull dimension greater than that of the original ring while the other has Krull dimension equal to that of the original ring. These examples are both taken from [38], Examples A and D.

Example 6.3.37. Let $R = k[\theta]$, where k is an algebraically closed field, and let $0 \neq q \in k$ such that $q^n \neq 1$ for any n . Then there is a q -skew derivation (σ, δ) on R such that $(\sigma, \delta) = (1, 0)$ on k and $\sigma(\theta) = q\theta$ and $\delta(\theta) = 1$. Let $S = R[x; \sigma, \delta]$. Then $\text{Kdim}(S) = \text{Kdim}(R) + 1$.

Proof. That (σ, δ) is a q -skew derivation is given by [38] Lemma 6.1. Consider a maximal ideal $\langle \theta - \alpha \rangle$ of R , where $0 \neq \alpha \in k$. Then $\sigma(\langle \theta - \alpha \rangle) = \langle \theta - \alpha/q \rangle$ and so $\langle \theta - \alpha \rangle$ is not stable under σ . Therefore, by Theorem 6.3.31, $\langle \theta - \alpha \rangle$ is quasistable. Thus, by Theorem 6.3.34,

$$\text{Kdim}(S) = \text{Kdim}(R) + 1.$$

□

Example 6.3.38. Let k, q, R and (σ, δ) be as in Example 6.3.37. Let $Q = \theta R$ and let T be the localisation of R at Q , so $T := R_Q$. Then (σ, δ) extends to T and $\text{Kdim}(T) = \text{Kdim}(T[x; \sigma, \delta])$.

Proof. The ring T is a commutative noetherian local ring with unique maximal ideal $P = \theta T$. Now, P is stable under σ and therefore (σ, δ) extends to T . However, $\theta \in P$ but $\delta(\theta) = 1 \notin P$ and P is not stable under δ and so P is not quasistable. Therefore

$$\text{Kdim}(T) = \text{Kdim}(T[x; \sigma, \delta]).$$

□

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